ON LOCATION AND APPROXIMATION OF CLUSTERS OF ZEROS: CASE OF EMBEDDING DIMENSION ONE

M. GIUSTI, G. LECERF, B. SALVY, AND J.-C. YAKOUBSOHN

ABSTRACT. Isolated multiple zeros or clusters of zeros of analytic maps with several variables are known to be difficult to locate and approximate. This article is in the vein of the α-theory, initiated by M. Shub and S. Smale in the beginning of the eighties. This theory restricts to simple zeros, i.e., where the map has corank zero. In this article we deal with situations where the analytic map has corank one at the multiple isolated zero, which has embedding dimension one in the frame of deformation theory. These situations are the least degenerate ones and therefore most likely to be of practical significance. More generally, we define clusters of embedding dimension one. We provide a criterion for locating such clusters of zeros and a fast algorithm for approximating them, with quadratic convergence. In case of a cluster with positive diameter our algorithm stops at a distance of the cluster which is about its diameter.

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Introduction

Foreword. In both theory and practice, dealing with multiple zeros of multivariate analytic maps remains a challenging problem. The now classical α-theory developed by M. Shub and S. Smale restricts to simple zeros. At such zeros the Jacobian is invertible, so the map has corank zero in our context, and this property remains of course valid in an open neighborhood. In this article we treat the next case, i.e., where the corank of the analytic map is 1 at the multiple zero. Note that the corank can drop in a neighborhood, where consequently it is *at most one*.

To avoid confusions, note that by *corank* of an analytic map at a multiple zero we mean the corank of its Jacobian, while by corank of a function at a critical point it is understood in singularity theory the corank of its Hessian (see [3]).

The corank 1 condition implies, through the implicit function theorem, that the zero lies on a smooth curve, hence its embedding dimension is 1. In the context of numerical analysis, the right issue is to isolate and approximate clusters of zeros, simple or not. Therefore, the right hypothesis is not the corank at most 1 condition at each zero of the cluster, but the embedding dimension 1 condition, as we exemplify in the next paragraphs. Our generalization of the α-theory is done under the latter hypothesis, which actually covers a large class of singularities encountered in practice.

Clusters of Embedding Dimension One. Formally speaking, by *cluster* we only mean *set*. Its *diameter* is the maximum distance between any two of its points. In the context of clusters of zeros, this word is convenient to refer to a set of zeros whose diameter is small compared to the distance to other zeros.

Let f be a complex analytic map defined on a connected open subset U of $\mathbb{C}^n$ to $\mathbb{C}^n$, where $\mathbb{C}^n$ is endowed with the canonical Hermitian dot product. A set of zeros of f is said to be a *cluster of zeros of embedding dimension* 1 if there exist two vector subspaces S and T of $\mathbb{C}^n$ of codimension 1, a ball in U containing the cluster, such that $\mathrm{Pr}_T \circ Df(x)$ is invertible on S, for all x in the ball, where Pr_T denotes the orthogonal projection to T. Note that this definition readily extends the notion of a multiple zero of corank 1 to the context of numerical analysis.

In this situation, we can choose orthonormal bases at the source and target of f, $(x, y) = (x_1, \ldots, x_{n-1}, y)$ and $(f_1, \ldots, f_{n-1}, \mathtt{g}) = (\mathtt{f}, \mathtt{g})$, such that the partial derivative of $\mathtt{f} = (f_1, \ldots, f_{n-1})$ with respect to the variables x is invertible in a ball centered at a zero ζ of f. Let ζ_x and ζ_y respectively denote the x and y coordinates of ζ. The implicit function theorem implies that there exists an analytic function $\phi(y) : W \to \mathbb{C}^{n-1}$ defined on a neighborhood W of ζ_y such that $\phi(\zeta_y) = \zeta_x$ and $\mathtt{f}(\phi(y), y) = \mathtt{f}(\zeta) = 0$ holds in W. Up to restricting W, $h(y) := \mathtt{g}(\phi(y), y)$ is well defined on W. We call it the *univariate reduction* of f. Looking for zeros of f is then reduced to looking for zeros on a smooth curve, and, after parametrization, for zeros of a univariate analytic function: its reduction. This motivated us to use this embedding dimension terminology. The reduction can also be viewed as an eliminating object (of the variables x) between the components of f, in a local analytic setting.

Note that multiple zeros of corank 1 are clusters of embedding dimension 1. In addition, this embedding dimension 1 property is preserved under small deformations (by *deformation*, we refer in this context to the classical theory given for example in [3]). The methods presented here for cluster location and approximation are not designed to treat *all* clusters of embedding dimension 1 but only the ones that are sufficiently *small* in some sense we will precisely quantify.

Example 1. The first example of a cluster of embedding dimension 1 is an ordinary quadratic point (zero of multiplicity 2) in any ambient space, which is also known as "a simple double point" (see [9]). There exists local coordinates in which it can be described by the map $(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^2)$, which is obviously of corank 1 at the origin. In Arnold's classification of singularities [3], this is the singularity A_1.

Example 2. More generally, the common archetype is the map

$$(x_1, \ldots, x_{n-1}, y) \mapsto (x_1, \ldots, x_{n-1}, y^m),$$

which admits the origin as a zero of multiplicity m and corank 1. Actually all maps with a multiple zero of corank 1 and multiplicity m are analytically equivalent to this example (called A_{m-1} in Arnold's classification of singularities [3]).

As an illustration of the deformation remark above, note that all small deformations of this fat point, described by the semi-universal deformation $y^m + \sum_{i=2}^{m} a_{m-i} y^{m-i}$ of y^m, keep the corank *at most* one property at any zero and yield a family of examples of clusters of embedding dimension 1, with m zeros, in an appropriate ball.

Example 3. Our third example of a cluster of embedding dimension 1 is a fat point of multiplicity 4 and corank 1. Let us consider an ellipse and one of its superosculating circles. They intersect at such a point: coordinates can easily be found when the situation is described by the equations $x_1^2 - 4x_1 + 4y^2 = 0$ and $x_1^2 - x_1 + y^2 = 0$ (here $n = 2$).

Counterexample. Let $f := (x_1, y) \mapsto (x_1^2 - e, y^2 - e)$, for a positive real parameter e. Consider the cluster composed of the four zeros of f. Since the origin belongs to the convex hull of this cluster, any ball containing the cluster also contains the origin. Although the corank of f is clearly at most one at each zero, it follows that this cluster cannot have embedding dimension 1 since $Df(0, 0) = 0$,

Notations. We introduce the basic definitions and conventions used all along this text. For any $\zeta \in \mathbb{C}^n$, and any real number $r \geq 0$, we use the following usual notation for balls: $B(\zeta, r) := \{x \in \mathbb{C}^n,\ \|x - \zeta\| < r\}$ and $\bar{B}(\zeta, r) := \{x \in \mathbb{C}^n,\ \|x - \zeta\| \leq r\}$.

More on Embedding Dimension One. All along this text, we assume that the map f is already given with well suited sets of coordinates (x, y) and $(\mathtt{f}, \mathtt{g})$ that satisfy the above properties. If this were not the case it would suffice to change the coordinates.

Something could be puzzling in the above definitions of embedding dimension 1 and reduction, in the context of clusters of zeros of positive diameter (i.e., not reduced to a multiple zero). These definitions seem to depend on a particular zero: actually this is not the case. Instead of considering only the implicit function defined when $\mathtt{f}$ equals 0, we introduce the following map Σ that allows one to handle the family of implicit functions defined when $\mathtt{f}$ equals a certain parameter that varies in a neighborhood of 0. Namely, for any $(x_0, y_0) \in U$ such that $D_x\mathtt{f}(x_0, y_0)$ is invertible, we introduce

$$\begin{array}{rrcl} \Sigma(\mathtt{f}, x_0, y_0; x, y): & U & \to & \mathbb{C}^n \\ & (x, y) & \mapsto & (D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x, y), y). \end{array}$$

Note that this map is classical for proving the equivalence between the local compositional inverse function and the implicit function theorems. The map Σ is invertible in a neighborhood of (x_0, y_0); its inverse is denoted by $\Phi(\mathtt{f}, x_0, y_0; z, y)$ and is defined on a neighborhood of (z_0, y_0), where $z_0 := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x_0, y_0)$. We also introduce

$$h(f, x_0, y_0; z, y) := \mathtt{g}(\Phi(\mathtt{f}, x_0, y_0; z, y)).$$

For convenience, we also use the notation $\phi(\mathtt{f}, x_0, y_0; y)$ for representing the $n-1$ first coordinates of $\Phi(\mathtt{f}, x_0, y_0; 0, y)$, so

$$(\phi(\mathtt{f}, x_0, y_0; y), y) = \Phi(\mathtt{f}, x_0, y_0; 0, y)$$

holds. Then, in a certain neighborhood of (x_0, y_0) (that will be described precisely when needed) looking at the zeros of f, in other words the solutions of $\mathtt{f}(x, y) = \mathtt{g}(x, y) = 0$, is equivalent to

$$h(f, x_0, y_0; 0, y) = 0, \quad x = \phi(\mathtt{f}, x_0, y_0; y),$$

hence reduces to considering a univariate equation.

By construction, we have $f(\phi(\mathtt{f}, x_0, y_0; y), y) = 0$, hence $\phi(\mathtt{f}, x_0, y_0; y)$ and $h(f, x_0, y_0; 0, y)$ do not depend on (x_0, y_0) in a neighborhood of (x_0, y_0). In particular, ϕ actually represents the parameterization of the implicit function defined by $\mathtt{f} = 0$. Moreover if (x_0, y_0) is an isolated solution of $f = 0$ then its multiplicity equals the multiplicity of y_0 as a solution of $h(f, x_0, y_0; 0, y) = 0$. In many places, we will write $h(f, x_0, y_0; z, .)$ for this single complex variable function of y, for fixed z.

Deflated Maps. For any $l \geq 0$, the lth *deflated function* of a univariate complex function h is nothing else than its lth derivative $h^{(l)}$. For any $l \geq 0$, all along this text, we study the lth *deflated map* $f^{[l]} = (\mathtt{f}, \mathtt{g}^{[l]})$ obtained from $f = (\mathtt{f}, \mathtt{g})$ according to the following recursive definition:

$$\mathtt{g}^{[0]} = \mathtt{g}, \quad \mathtt{g}^{[l+1]} = \frac{\det(D(\mathtt{f}, \mathtt{g}^{[l]}))}{\det(D_x\mathtt{f})}, \quad \text{for } l \geq 0, \tag{1}$$

where det denotes the determinant map, the implicit basis that we consider is the canonical one given by (x, y).

Point Estimates. Let f denote again an analytic map from an open subset U of $\mathbb{C}^n$ to $\mathbb{C}^n$ and $a \in U$. Then its kth derivative $D^k f(a)$ belongs to the space $\mathcal{L}_k(\mathbb{C}^n; \mathbb{C}^n)$ of $\mathbb{C}$-multilinear maps from k copies of $\mathbb{C}^n$ to $\mathbb{C}^n$. For any such multilinear map L we use the classical *norm*:

$$\|L\| := \sup_{\substack{u_1 \in \mathbb{C}^n, \dots, u_k \in \mathbb{C}^n \\ \|u_1\| = \cdots = \|u_k\| = 1}} \|L(u_1, \dots, u_k)\|.$$

Concerning Newton's operator, we will use the following notation:

$$N(f; x) := x - Df(x)^{-1} f(x),$$

and for Schröder's operator [43] (if h is a univariate map) we write:

$$N_m(h; x) := x - m \frac{h(x)}{h'(x)}.$$

Of course, for analytic functions, N_1 and N coincide.

So called *point estimates* are quantities defined from norms of differential maps at a given point. Three important such quantities are used for simple zeros: γ, β and α. The first one, namely $\gamma(f; a)$, helps control the function locally:

$$\gamma(f; a) := \sup_{k \geq 2} \left\| Df(a)^{-1} \frac{f^{(k)}(a)}{k!} \right\|^{\frac{1}{k-1}}.$$

In particular, the radius of convergence of the power series expansion of f at a is at least $1/\gamma$. The second quantity is the length of the Newton iteration step: $\beta(f; a) := \left\| Df(a)^{-1} f(a) \right\|$; the third one is their product $\alpha(f; a) := \beta(f; a)\gamma(f; a)$.

For a univariate function h, in order to deal with clusters of m zeros, counting multiplicities, and multiple zeros of multiplicity m, the previous quantities are generalized as follows, for any $l \in \{0, \dots, m-1\}$ (which will have to do with a level l of deflation):

$$\gamma_m(h; a) := \sup_{k \geq m+1} \left(\frac{m! |h^{(k)}(a)|}{k! |h^{(m)}(a)|} \right)^{\frac{1}{k-m}},$$

$$\beta_{m,l}(h; a) := \sup_{l \leq k \leq m-1} \left(\frac{m! |h^{(k)}(a)|}{k! |h^{(m)}(a)|} \right)^{\frac{1}{m-k}},$$

$$\alpha_{m,l}(h; a) := \gamma_m(h; a) \beta_{m,l}(h; a).$$

As expected, these quantities coincide with the previous ones when letting $m = 1$ and $l = 0$. For short we let $\beta_m := \beta_{m,0}$ and $\alpha_m := \alpha_{m,0}$. If $\gamma_m(f; a) = 0$ then we consider $1/\gamma_m(f; a) = +\infty$, as a natural convention. Together with these quantities, the following auxiliary function comes naturally:

$$\psi_m(u) := 2(1-u)^{m+1} - 1.$$

In short we shall also write $\psi(u)$ for $\psi_1(u) = 1 - 4u + 2u^2$.

Summary of Our Contributions. We now present a summary of our main contributions section by section. We explain how the paper is organized, and how the results are connected.

α-Theory for Simple Zeros. Our first section is devoted to prerequisites on majorant series, and to classical results on location and approximation of simple zeros, following the now classical α-theory. We provide a new synthetic shorter presentation of this theory.

The first properties of majorant series are recalled in the first subsection. The next two subsections are respectively devoted to Pellet's location criterion of simple zeros, and to the calculation of upper bounds on a point estimate from the same estimate known at a close point.

The fourth subsection gives a short proof of the γ-theorem (namely, Theorem 1.16), that quantifies the convergence of Newton's operator in a neighborhood of a simple zero, from point estimates at the zero. Then we explain how these estimates can be approximated from the same estimates at any given point located sufficiently close to the zero. The combination of the γ-theorem with the location criterion thus leads us to a "weak" version of the α-theorem. By contrast with the γ-theorem, the α-theorem quantifies the convergence of Newton's operator from estimates at the initial point of the Newton iteration. The α-theorem is often more relevant to practical concerns.

A "strong" version of the α-theorem (Theorem 1.17) is then established in the fifth subsection. In the vein of Kantorovich's analysis, we generalize the α-theorem given by Wang and Han in [55] (see Corollary 1.18), in order to show that, if Pellet's location criterion is satisfied at a given point, then the Newton iterates of this point converge to a simple zero.

The first section ends with a quantitative version of the implicit function theorem (Theorem 1.19), that is a crucial ingredient to handle the univariate reduction h in practice. This result is not new: it is a consequence of the α-theorem (Corollary 1.18), and of the results given by Dedieu *et al.* in [8, Section 3]. In Appendix A, we generalize the latter results to geometric majorant series. More precisely, we provide a sharper geometric series majoration of the compositional inverse map, in cases where the derivative at the given point is different from identity. These sharper results are not used outside of this appendix, but they are useful to tune the algorithms in practice.

Reduction to One Variable. Section 2 gathers technical results needed by the location and the approximation algorithms.

For the sake of completeness, in the first subsection, we recall the location criterion given in [12] for univariate maps, and that generalizes Pellet's criterion to clusters of zeros. Briefly speaking, for a univariate function h, if $\alpha_{m,l}(h;a)$ is sufficiently small, then $h^{(l)}$ admits a cluster of $m-l$ zeros in a ball centered at a of radius about $\beta_{m,l}(h;a)$. In addition, if a lies in the convex hull of this cluster then the diameter of the cluster is also about $\beta_{m,l}(h;a)$. This crucial observation allows one to compute approximations of diameters of clusters. When we say that a point lies far from or close to the cluster, we refer to its diameter as the implicit scale.

The second subsection contains an algorithm to compute approximations of the function $\beta_{m,l}$. This device will only be used by the approximation algorithm in Section 4.

In the third subsection, we study the interplay between the deflated map and its deflated univariate reduction. More precisely we prove the following relation:

$$D_y^l h(f, x_0, y_0; z, y) = \mathtt{g}^{[l]} \circ \Phi(\mathtt{f}, x_0, y_0; z, y), \tag{2}$$

which provides an efficient way of computing Taylor expansions of h.

In the last subsection, we study how $\beta_{m,l}(h(z,.); y)$ and $\gamma_m(h(z,.); y)$ vary in z when y is fixed. The common archetype the reader may keep in mind is $h(z,y) = zy^l + y^m$: the diameter of the cluster of zeros of $D_y^l h(z,.) = 0$ varies like $\|z\|^{1/(m-l)}$. Therefore, in order to approximate $D_y^l h(0,y)$ at a usable scale, $\|z\|^{1/(m-l)}$ will be required to be sufficiently small, which is a key observation for our algorithms.

Cluster Location. Cluster location is addressed in Section 3, and means finding *(a)* a point; *(b)* an estimate for the radius of a ball centered at this point and containing zeros of f, or a deflated map of f; *(c)* the radius of a zero-free region beyond this ball. Our Theorem 3.1 generalizes the location criterion given in [12, Section 1], that restricts to univariate functions. In addition, Theorem 3.1 is more general than the location criterion of Dedieu and Shub presented in [9], that deals with our present case of study but restricting to clusters containing 2 zeros (that is $m = 2$ in our context).

Let us summarize the main feature of Theorem 3.1 in an informal way. Let (x_0, y_0) be a given point, and let $m \geq 1$ and $l \in \{1, \ldots, m-1\}$ be given integers. Let κ denote the first integer such that $2^\kappa \geq m - l$, let $x_0' := N^\kappa(\mathtt{f}(., y_0); x_0)$, and let $z_0' := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x_0', y_0)$. Theorem 3.1 can be seen as a sufficient criterion that decides the existence of a cluster of $m - l$ zeros of the lth deflated map $f^{[l]}$, from the sole knowledge of point estimates of g and $\Sigma(\mathtt{f}, x_0, y_0; ., .)$ at (x_0, y_0), and of $h(f, x_0, y_0; ., .)$ at (z_0', y_0). This criterion is proved to be necessary for sufficiently small clusters, when (x_0, y_0) is sufficiently close to the cluster. In addition, the point estimates involved in Theorem 3.1 are computable for several classes of maps f. Numerical experiments with Theorem 3.1 are provided in Section B.2 (of the appendix) for polynomial maps.

Cluster Approximation. For any given integers m, l and l' such that $m \geq 1$, $l \in \{0, \ldots, m-1\}$, and $l' \in \{0, \ldots, l\}$, we wish to approximate a cluster with $m - l'$ zeros of the l'th deflated map $f^{[l']}$ by means of the lth deflated map $f^{[l]}$. The initial point of this approximation process will always be written (x_0, y_0). This problem is solved in Section 4.

Section 4 starts with the definition of the operator $N_{m,l,l'}$, that contains branching depending on the real parameters r_y and $\mathcal{G}_y$. Roughly speaking, this operator does the following computations when applied to (x_0, y_0): it first computes (x_0', y_0) such that $z_0' := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x_0', y_0)$ is of order $2(m - l')$; then it computes the Schröder iterate y_0' of y_0 as $y_0' := N_{m-l}(D_y^l h(f, x_0, y_0; z_0', .); y_0)$. Depending on the location of y_0 with respect to the cluster of zeros of $D_y^{l'} h(f, x_0, y_0; z_0', .)$, the output (x_1, y_1) of the operator is computed from y_0 or from y_0'. The latter choice involves the use of the parameters r_y and $\mathcal{G}_y$.

Then we exhibit a stopping criterion to ensure that the iterates of (x_0, y_0) by $N_{m,l,l'}$ stop close to the cluster of $f^{[l']}$. This criterion involves a third parameter $\mathcal{G}_z$. The combination of the operator $N_{m,l,l'}$ together with the stopping criterion is called the *approximation algorithm* in the sequel.

When using our algorithm with $l = l' = 0$, the computation of y_0' only requires the first derivative of $h(f, x_0, y_0; z_0', .)$. On the opposite, with $l = m - 1$ and $l' = 0$, our algorithm is close to performing the Newton iteration on the *full* deflated map. Our stopping criterion then allows one to stop this iteration close to the cluster of the original system. These extreme cases motivate this unified presentation in terms of l and l'.

In Theorem 4.1, if (x_0, y_0) is sufficiently close to a cluster with $m-l'$ zeros of the l'th deflated map, and if this cluster is sufficiently small, then we give formulas to compute suitable values of the parameters r_y, $\mathcal{G}_y$ and $\mathcal{G}_z$ such that the approximation algorithm is well defined and produces a sequence of iterates that converge quadratically to the cluster, and that stops at a distance to the cluster which is of the order of magnitude of its diameter. Furthermore, we precisely quantify what is meant by "sufficiently close" and by "sufficiently small" in terms of point estimates of $\mathtt{f}$, $\mathtt{g}$, $\Sigma(\mathtt{f}, x_0, y_0; ., .)$, and $h(f, x_0, y_0; ., .)$ at a point of the cluster. We also give a precise statement for the "quadratic convergence" and for the "order of magnitude" of the distance to the cluster. The proof of Theorem 4.1 is quite long (it occupies subsections 4.2 to 4.8), and it relies strongly on the univariate approximation algorithm presented in [12, Theorem 4.5].

Theorem 4.1 can be seen as a generalization of the γ-theorem to clusters of embedding dimension 1, since it involves points estimates at the cluster. In practice, these estimates are of course unknown, and we wish to have a result similar to the α-theorem, which only requires point estimates at the initial point (x_0, y_0). In the last subsection of Section 4, with an algorithmic presentation, we explain how one can achieve this goal in a similar way as we did in Section 1.4 to obtain our "weak" version of the α-theorem for simple zeros. More precisely, we put together the location criterion of Theorem 3.1 with the algorithm stated in Theorem 4.1.

The approximation algorithm of this paper generalizes the one variable approximation algorithm presented in [12, Section 4]. It can be used for wide classes of functions for which the necessary point estimates are computable. In Appendix B, we report on our implementations of the location and the approximation algorithms with polynomial maps. These experiments are in good agreement with our theoretical analysis.

Related Work. In the following paragraphs, we discuss other approaches to cluster detection and to generalizations of the Newton iteration. Recall that the univariate case that we generalize here is treated in [12], to which we refer for the bibliography. Here, we focus on several variables maps.

α-Theory. In dimension larger than one and in the spirit of the α-theory developed by Shub and Smale in [46, 44, 45, 47, 4], the only quantitative result generalizing simple zeros is due to Dedieu and Shub. In [9], they present a qualitative version of Rouché's theorem that guarantees the existence of clusters of two zeros. In this article, we present a more general criterion

that deals with higher multiplicities. We also provide an approximation algorithm based on Schröder's operator [43, 53]. As for simple zeros, all our results are effective and certified, as exemplified in Appendix B.

In the next paragraphs, we present other works concerning approximation of multiple zeros, that are outside the scope of the α-theory. This presentation is not intended to be exhaustive but tends to reflect some mainstreams among a vast amount of literature, to which our results are most related.

Deflation. In the multivariate case, multiple zeros can be approximated by deflation algorithms: they mainly consist in differentiating well chosen equations, according to the nature of the singularity. Ojika, Watanabe, Mitsui [36] and then Ojika [35] proposed a so-called modified deflation algorithm based on hybrid symbolic and numerical computations. In [32], this technique is improved and the number of deflation stages is proved to be bounded by the multiplicity. The questions of complexity and stability of these mixed approaches have not been studied yet. On the other hand, in the computer algebra framework (that concerns non archimedian valuated fields) a general deflation algorithm has been described by Lecerf in [31]: this gives an algorithm for quadratic approximation of multiple zeros in time which remains polynomial in the multiplicity. Yet this method has not led to numerical experiments.

The present work deals with a particular case of deflated systems of *depth one*, according to the terminology of [31]. The main difference is that we deal with clusters instead of multiple zeros, which turns out to be much more difficult. Dealing with clusters of embedding dimension higher that 1 by means of deflated systems of depth higher than one is a difficult challenge.

Corrected Newton Methods. Experimentally, if Newton's method converges to an isolated singularity then the convergence is linear. A quantitative analysis of this property is complicated. Several authors have contributed to this topic. Motivated by Ostrowski [37], Rall studied some particular cases in [40]. Then, based on Reddien's advances [41, 42], Decker and Kelley [6, 7] precise the convergence rate for singular problems of first and second orders for maps between Banach spaces. Griewank and Osborne [14, 16, 15] propose generalizations and precise convergence domains.

In the univariate case, corrected operators are well known from Schröder's work [43] and have been improved by Van de Vel [53, 54]. Recently, this approach has been generalized to Pham systems [26].

In general, correcting Newton's iteration to reestablish quadratic convergence has been studied in numerous works. In this vein, an important class of approximation methods is based on a so called *regularization* of the Newton correction: we refer to [2, Theorem 1] for recent advances and references on this subject. In fact, the construction described therein is obtained from the one we mention in the next paragraph.

Bordering Techniques. So-called bordering techniques form a large class of methods to deal with singularities. As for the deflation techniques, the main idea is the construction of a system that admits a simple solution in place of the singular solution of the original system, hence Newton's operator can be used. For *double* zeros, this technique is explored in [56, 57, 15, 22, 23, 33]. For multiple zeros with corank 1, Tsuchiya proposes a method in [52], that is based on [57].

Augmented Systems. The construction of *augmented systems* can also be used to approximate isolated singular zeros. Roughly speaking, the basic idea consists in introducing new variables corresponding to coordinates of vectors belonging to some kernels of well chosen linear maps constructed from derivatives of the given map and depending on the type of the singularity.

The number of articles dealing with these techniques is too huge to cite them all, we refer to [17, 39, 18, 13] for historical details and references. The most general method is exposed by Kunkel in [30] (as a generalization of [28, 29]) and is extended to Banach spaces in [20].

Cluster Approximation. Among all these techniques, the cluster approximation problem remains unexplored: relating the zeros of bordered, augmented or deflated systems to the ones of the original problem is still an open question in full generality. For the first time, we solve this problem here for clusters of embedding dimension 1. Our use of the implicit function theorem by means of truncated power series computations is a major difference compared to the other aforementioned methods. In particular, we do not introduce extra unknowns at the opposite of the construction of augmented systems.

Algebraic Topology. Far from our present concerns, a vast amount of results for locating zeros of analytic maps are designed from numerical integration and residue formula. We refer to [27] for an exposition of some of these results and for an historical presentation. Other original zero counting related methods are based on topological degree theory [51, 25].

Global Techniques. Lastly, from a global point of view, it is worth mentioning that several techniques for polynomial system solving are robust in presence of multiplicities and clusters, but their complexities take into account all the zeros of the systems. Originating from commutative algebra, let us mention recent advances in Gröbner basis computation [10, 11] and geometric solving [31]. From a numerical point of view, formal computations can be used as a preprocess: for instance geometric solving brings back to solving a univariate polynomial, or suitable constructions bring back to linear algebra, as in [5]. Pure numerical techniques mostly rely on homotopy continuation, as in [34, 50, 48, 49], for example.

1. α-Theory for Simple Zeros

For the sake of completeness this first section gathers material of α-theory for simple zeros. We start with Pellet's location criterion, that we directly prove via Rouché's theorem. Then we recall the γ-theorem, generalize [55] to majorant series, prove the α-theorem, and recall the quantitative version of the local compositional inverse theorem.

1.1. **Majorant Series.** We briefly recall the main basic properties of majorant series and geometric majorant series. We refer to [12, Appendix A] for proofs. We denote by $\mathbb{R}\{t\}$ the algebra of real power series with positive radius of convergence. We say that a series $\lambda t/(1-\rho t)$ is a *geometric series*, for the sequence of its coefficients is in geometric progression. We widely use the following notation for the exponential generating series of the norms of the derivatives of f at $a \in \mathbb{C}^n$:

$$[f]_a := \sum_{k\geq 0} \|D^k f(a)\| \frac{t^k}{k!} \in \mathbb{R}\{t\}.$$

We consider the following partial order $\leq$ over $\mathbb{R}\{t\}$. Let F and G be in $\mathbb{R}\{t\}$, we write $F \leq G$ when $F^{(k)}(0) \leq G^{(k)}(0)$ for all $k \geq 0$. Then we say that a power series $F \in \mathbb{R}\{t\}$ is a *majorant series* for an analytic map f at a point a if $[f]_a \leq F$.

Proposition 1.1. *The partial order on $\mathbb{R}\{t\}$ satisfies the following compatibility rules:*

a. *For all nonnegative x in $\mathbb{R}$, $x \geq 0$, seen as the constant function in $\mathbb{R}\{t\}$;*
b. *For all F in $\mathbb{R}\{t\}$, $F \geq 0$ is equivalent to $-F \leq 0$;*
c. *For all F, G and H in $\mathbb{R}\{t\}$, if $F \leq G$ then $F + H \leq G + H$;*
d. *For all F, G and H in $\mathbb{R}\{t\}$, if $F \leq G$ and $H \geq 0$ then $FH \leq GH$.*

Let f denote an analytic map from a connected open subset $U \subseteq \mathbb{C}^n$ to $\mathbb{C}^m$. The map $[.]$ satisfies the following basic properties:

Proposition 1.2. *According to the above notation, for any $a \in U$ we have:*

a. $[f]_a \geq 0$;
b. $[f]_a = 0$ *is equivalent to $f = 0$ in a neighborhood of a;*
c. $[cf]_a = |c| [f]_a$, *for all $c \in \mathbb{C}$;*
d. $[f + g]_a \leq [f]_a + [g]_a$.

Next follows a list of less basic properties. Let $a \in U$, F be such that $[f]_a \leq F$.

Proposition 1.3. *(Differentiation)* $[Df]_a \leq F'$.

Let $b \in U$ be such that $\|b - a\|$ is less than the radius of convergence of F.

Proposition 1.4. *(Evaluation)* $\|f(b)\| \leq F(\|a - b\|)$.

Proposition 1.5. *(Translation)* $[f]_b \leq F(t + \|a - b\|)$.

Corollary 1.6. *(Translation for geometric series) If $F = F(0) + \frac{\lambda t}{1-\rho t}$, then*

$$[f - f(b)]_b \leq \frac{\lambda' t}{1 - \rho' t},$$

with $\lambda' := \lambda/(1 - \rho\|b - a\|)^2$ and $\rho' := \rho/(1 - \rho\|b - a\|)$.

Proposition 1.7. *(Product) Let f be an analytic map from U to $\mathcal{L}(\mathbb{C}^n; \mathbb{C}^m)$ and g be an analytic map from U to $\mathcal{L}(\mathbb{C}^p; \mathbb{C}^n)$, then we have:*

$$[h]_a \leq [f]_a [g]_a,$$

where h is defined by

$$\begin{aligned} h : U &\to \mathcal{L}(\mathbb{C}^p; \mathbb{C}^m) \\ x &\mapsto f(x) \circ g(x). \end{aligned}$$

In particular, it follows that the product of majorant series of univariate functions is a majorant series for the product of these functions.

Proposition 1.8. *(Composition) Let g be an analytic map from $\mathbb{C}^m$ to $\mathbb{C}^p$, defined in the neighborhood of $f(a)$ and G be such that $[g]_{f(a)} \leq G$. Let $h := g \circ f$, then we have*

$$[h]_a \leq G \circ (F - F(0)).$$

Corollary 1.9. *(Composition for geometric series) According to the notation of the previous proposition, if $F = F(0) + \frac{\lambda_f t}{1-\rho_f t}$, $G = G(0) + \frac{\lambda_g t}{1-\rho_g t}$, then*

$$[h - h(a)]_a \leq \frac{\lambda t}{1 - \rho t}, \text{ where } \lambda := \lambda_f \lambda_g, \ \rho := \rho_f + \lambda_f \rho_g.$$

In addition, $f(B(a, 1/\rho)) \subseteq B(f(a), 1/\rho_g)$.

Lastly, we recall a formula concerning inversion of linear maps.

Proposition 1.10. *(Inversion) Let U be an open neighborhood of a in $\mathbb{C}^n$ and f be analytic from U to $\mathcal{L}(\mathbb{C}^m; \mathbb{C}^m)$ such that $f(a) = \mathrm{Id}$. Let $F \in \mathbb{R}\{t\}$ such that $[f]_a \leq F$, then*

$$\left[f^{-1}\right]_a \leq \frac{1}{1 + F(0) - F}.$$

In addition, the radius of convergence of $\frac{1}{1+F(0)-F(t)}$ is at least

$$\bar{\rho} := \sup \left(s < \rho \mid 1 + F(0) - F(r) > 0, \text{ for all } r \in [0, s]\right),$$

where ρ denotes the radius of convergence of F.

1.2. **Location of Simple Zeros.** We show that the optimal location criterion (in terms of the quantity α) given by Wang and Han in [55] can be deduced from Rouché's theorem. We start with the following proposition, that generalizes Pellet's criterion [38] to simple zeros of several variables maps. We follow the same presentation as in [12, Section 1].

For the rest of this section, f denotes an analytic map from a connected open subset $U \subseteq \mathbb{C}^n$ to $\mathbb{C}^n$, x_0 belongs to U and $Df(x_0)$ is assumed to be invertible. For short we let $\alpha := \alpha(f; x_0)$, $\beta := \beta(f; x_0)$, $\gamma := \gamma(f; x_0)$. For a majorant series F, we will often use the notation

$$\tilde{F} := F - (1 + F'(0))t.$$

Proposition 1.11. *Let $F \in \mathbb{R}\{t\}$ be such that $\left[Df(x_0)^{-1} f\right]_{x_0} \leq F$. Let $r > 0$ be a real number smaller than the radius of convergence of F such that $\bar{B}(x_0, r) \subseteq U$ and*

$$\tilde{F}(r) < 0 \tag{3}$$

then f has exactly one simple zero in $\bar{B}(x_0, r)$.

Proof. First observe that it is not restrictive to assume $Df(x_0) = \mathrm{Id}$. We introduce $g : U \to \mathbb{C}^n$, $g(x) := f(x) - f(x_0)$. Let $w \neq x_0$ be such that $s := \|w - x_0\| \leq r$. By Taylor expansion, we have $g(w) = w - x_0 + \mathcal{O}_{x_0}(w - x_0)^2$, then by the triangular inequality, we get

$$\frac{\|g(w)\|}{s} \geq 1 - \sum_{j \geq 2} \frac{F^{(j)}(0)}{j!} s^{j-1} \geq 1 - \frac{F(r) - F(0) - rF'(0)}{r} > \frac{F(0)}{r},$$

where the latter inequality follows from (3). Here $\mathcal{O}_{x_0}$ represents the classical Landau "big O" notation in the neighborhood of x_0. As a consequence, x_0 is the only simple zero of g in the ball $\bar{B}(x_0, r)$. Moreover, when $s = r$, the inequality above yields $\|f(w) - g(w)\| < \|g(w)\|$. In particular, $g(w)$ does not vanish and therefore the multivariate generalization of Rouché's theorem [1, Chapter 1, Theorem 2.5] asserts that f and g have the same number of zeros in $\bar{B}(x_0, r)$, counting multiplicities. ☐

We now examine the important special case of the majorant series $F = F(f, x_0; t)$, defined by

$$F(f, x_0; t) := \beta(f; x_0) + \frac{t}{1 - \gamma(f; x_0)t}, \text{ since } \left[Df(x_0)^{-1} f\right]_{x_0} \leq F(f, x_0; t).$$

Inequality (3) rewrites

$$\tilde{F}(f, x_0; r) := \beta - r + \frac{\gamma r^2}{1 - \gamma r} < 0. \tag{4}$$

Assuming $\gamma r < 1$ and $\bar{B}(x_0, r) \subseteq U$, the previous proposition specializes to: if $\tilde{F}(f, x_0; r) < 0$ then f has exactly one simple zero in $\bar{B}(x_0, r)$.

$\tilde{F}(f, x_0; r)$ is convex on the range $0 \leq r < 1/\gamma$, and inequality (4) admits solutions in this range if and only if $\alpha < 3 - 2\sqrt{2}$. In addition, by convexity, this set of solutions forms a range that we write $(r^-(f; x_0), r^+(f; x_0))$, where $r^-(f; x_0)$ and $r^+(f; x_0)$ (that may be infinity) are the roots of $\tilde{F}(f, x_0; r) = 0$. For short, we let $r^- := r^-(f; x_0)$ and $r^+ := r^+(f; x_0)$ and a direct calculation yields:

$$r^- := \beta \frac{2}{1 + \alpha + \sqrt{1 - 6\alpha + \alpha^2}}, \quad r^+ := \frac{1 + \alpha + \sqrt{1 - 6\alpha + \alpha^2}}{4\gamma}. \tag{5}$$

If $\alpha \neq 0$ then the following inequality chain follows from the facts that r^- is convex increasing while r^+ is concave decreasing as functions of α:

$$\beta < (1 + \alpha)\beta < r^- < \left(1 + \left(2 + 3\frac{\sqrt{2}}{2}\right)\alpha\right)\beta < \left(1 + \frac{\sqrt{2}}{2}\right)\beta < \tag{6}$$

$$\frac{1 - \frac{\sqrt{2}}{2}}{\gamma} < \frac{1 - (1 + \sqrt{2})\alpha}{2\gamma} < r^+ < \frac{1 - \alpha}{2\gamma} < \frac{1}{2\gamma}. \tag{7}$$

We summarize this discussion in the following theorem.

Theorem 1.12. [55, Proposition 2] *If $\alpha < 3 - 2\sqrt{2}$ then for any r such that $r^- \leq r < r^+$ and $\bar{B}(x_0, r) \subseteq U$, f has exactly one simple zero in $\bar{B}(x_0, r)$.*

It is worth mentioning that this result is sharp, for it applies to $\tilde{F}(f, x_0; r)$, seen as a univariate map of r: at $r = 0$, it is easy to see $\beta(\tilde{F}(f, x_0; .); 0) = \beta$ and $\gamma(\tilde{F}(f, x_0; .); 0) = \gamma$. In practice, the next corollary will reveal easier to use:

Corollary 1.13. *If $\alpha < 3 - 2\sqrt{2}$ and $\bar{B}(x_0, (1 + \sqrt{2}/2)\beta) \subseteq U$ then f has exactly one simple zero in both $\bar{B}(x_0, (1+\sqrt{2}/2)\beta)$ and $B(x_0, (1-\sqrt{2}/2)/\gamma) \cap U$.*

Proof. We apply the previous theorem with the analytic extension of f on $B(x_0, 1/\gamma)$, for $r = (1 + \sqrt{2}/2)\beta$ (according to (6)) and then for any r in a left neighborhood of $(1 - \sqrt{2}/2)/\gamma$ (according to (7)). □

1.3. Translations of Point Estimates.

From estimates at a given point, getting upper bounds on the same quantities at another close point is a central operation. The following proposition will be used in proofs of the γ and α-theorems below.

Proposition 1.14. *Let F be such that $\left[Df(x_0)^{-1}f\right]_{x_0} \leq F$, then*

$$\left[Df(x_0)^{-1}Df\right]_{x_0} \leq F' \text{ and } \left[Df^{-1}Df(x_0)\right]_{x_0} \leq -\frac{1}{\tilde{F}'}.$$

Proof. Again we can assume $Df(x_0) = \mathrm{Id}$ and start with differentiating $[f]_{x_0} \leq F$: according to Proposition 1.3, we obtain $[Df]_{x_0} \leq F'$. Then the second majoration follows from Proposition 1.10. □

Translations of α, β and γ are useful for various tasks, we shall use them several times.

Proposition 1.15. [4, Chapter 8, Proposition 3] *Let $x_1 \in U$, $r := \|x_1 - x_0\|$ such that $u := \gamma r < 1 - \sqrt{2}/2$, then*

a. $\alpha(f; x_1) \leq \dfrac{\alpha(1-u)+u}{\psi(u)^2}$;

b. $\beta(f; x_1) \leq \dfrac{(1-u)}{\psi(u)}(\beta(1-u)+r)$;

c. $\gamma(f; x_1) \leq \dfrac{\gamma}{(1-u)\psi(u)}$;

d. $\|Df(x_1)^{-1}Df(x_0)\| \leq \dfrac{(1-u)^2}{\psi(u)}$;

e. $\|Df(x_0)^{-1}Df(x_1)\| \leq \dfrac{1}{(1-u)^2}$.

Proof. Using the previous proposition with $F = F(f, x_0; t)$, we find:

$$\left[Df(x_0)^{-1}Df\right]_{x_0} \leq \frac{1}{(1-\gamma t)^2} \text{ and } \left[Df^{-1}Df(x_0)\right]_{x_0} \leq \frac{(1-\gamma t)^2}{\psi(\gamma t)}.$$

Parts (d) and (e) directly follow via Proposition 1.4.

Then, we use the translation property for majorant series: according to Proposition 1.5, we have $\left[Df(x_0)^{-1}f\right]_{x_1} \leq F(f, x_0; r+t)$, hence

$$\begin{aligned}\left[Df(x_0)^{-1}f\right]_{x_1} &\leq \beta + \frac{r+t}{1-\gamma(r+t)} \\ &= \beta + \frac{r}{1-u} + \frac{1}{(1-u)^2}\frac{t}{1-\frac{\gamma}{1-u}t}.\end{aligned}$$

From $\left[Df(x_1)^{-1}f\right]_{x_1} \leq \|Df(x_1)^{-1}Df(x_0)\| \left[Df(x_0)^{-1}f\right]_{x_1}$, we deduce:

$$\left[Df(x_1)^{-1}f\right]_{x_1} \leq \frac{(1-u)^2}{\psi(u)}\left(\beta + \frac{r}{1-u} + \frac{1}{(1-u)^2}\frac{t}{1-\frac{\gamma}{1-u}t}\right),$$

from which directly follow parts (b) and (c), and finally (a). □

1.4. **γ-Theorem.** The γ-theorem quantifies the convergence of Newton's operator from point estimates at the zero. It is useful for homotopy continuation, when combined to the previous upper bounds on translation.

Theorem 1.16. [4, Chapter 8, Proposition 1] *Let $\zeta \in U$ and $r > 0$ be a real number such that $f(\zeta) = 0$, $Df(\zeta)$ is invertible, $\bar{B}(\zeta, r) \subseteq U$, $u := \gamma(f;\zeta)r < 1 - \sqrt{2}/2$ and $u/\psi(u) \leq 1$. Then, for any $x_0 \in \bar{B}(\zeta, r)$ the sequence $(x_k)_{k\in\mathbb{N}}$ recursively defined by $x_{k+1} := N(f; x_k)$ is well defined, has all elements belonging to $\bar{B}(\zeta, r)$ and*

$$\|x_k - \zeta\| \leq \left(\frac{u}{\psi(u)}\right)^{2^k-1} \|x_0 - \zeta\|, \text{ for all } k \geq 0.$$

Proof. A direct calculation gives:

$$\left[Df(\zeta)^{-1}(Df(x)x - f(x))\right]_\zeta \leq tF'(f,\zeta;t) - F(f,\zeta;t).$$

Using Propositions 1.14 and 1.7 then yields:

$$\begin{aligned}\left[x - Df(x)^{-1}f(x)\right]_\zeta &= \left[(Df(x)^{-1}Df(\zeta))(Df(\zeta)^{-1}(Df(x)x - f(x)))\right]_\zeta \\ &\leq -\frac{tF'(f,\zeta;t) - F(f,\zeta;t)}{\tilde{F}(f,\zeta;t)} = \frac{\gamma(f;\zeta)t^2}{\psi(\gamma(f;\zeta)t)}.\end{aligned}$$

By means of Proposition 1.4, evaluating this series majoration at $x = x_0$ and $t = r$ gives

$$\|x_1 - \zeta\| \leq \frac{\gamma(f;\zeta)}{\psi(u)}\|x_0 - \zeta\|^2 \leq \frac{u}{\psi(u)}\|x_0 - \zeta\| \leq \|x_0 - \zeta\|.$$

A straightforward induction concludes the proof. □

Combining the previous results actually leads to a weak α-theorem (see next subsection). More precisely, let $x_0 \in U$ be such that $Df(x_0)$ is invertible and $\bar{B}(x_0, (1+\sqrt{2}/2)\beta) \subseteq U$. Let $v := (1+\sqrt{2}/2)\alpha$ and assume $v < 1-\sqrt{2}/2$, which is equivalent to $\alpha < 3 - 2\sqrt{2}$. Then, from Corollary 1.13, there exists a simple zero ζ in $\bar{B}(x_0, (1+\sqrt{2}/2)\beta)$. Let $u := \frac{v}{(1-v)\psi(v)}$ and assume $u < 1 - \sqrt{2}/2$ and $\frac{u}{\psi(u)} \leq 1$, Proposition 1.15 gives:

$$\gamma(f;\zeta) \leq \frac{\gamma}{(1-v)\psi(v)},$$

hence $\gamma(f;\zeta)(1+\sqrt{2}/2)\beta \leq u$ and Theorem 1.16 asserts quadratic convergence from x_0. The supremum value $\hat{\alpha}$ of the α's satisfying these conditions is:

$$\hat{\alpha} := \sup\left(\frac{v}{1+\sqrt{2}/2} \mid 0 \leq v < 1-\sqrt{2}/2,\ u < 1-\sqrt{2}/2,\ \frac{u}{\psi(u)} \leq 1\right).$$

An easy calculation produces $0.06571 < \hat{\alpha} < 0.06572$. This is not the best possible value. Indeed the optimal condition on α yielding quadratic convergence of Newton's iterator is given in [55, Theorem 1]: the critical value is $3-2\sqrt{2} \approx 0.17157$. This is the next result we recall.

1.5. **α-Theorem.** The α-theorem below presents a huge interest in practice since it combines location and quantitative approximation from estimates at the initial point. As in [55], we use the dominating sequence technique, that consists in exhibiting an increasing sequence $(t_k)_{k\in\mathbb{N}}$ of nonnegative real numbers such that $\|x_{k+1}-x_k\| \leq t_{k+1}-t_k$, where x_k are the Newton iterates of x_0. This idea goes back to Kantorovich [24]. The next theorem shows that once Pellet's criterion is satisfied at x_0 then the Newton iterates of x_0 converge quadratically to a simple zero. We thus generalize the original idea of Wang and Han in [55] to majorant series. We do not use Proposition 1.11, but provide another proof that does not rely on Rouché's theorem.

Theorem 1.17. *Let $F \in \mathbb{R}\{t\}$ be such that $\left[Df(x_0)^{-1}f\right]_{x_0} \leq F$. Let $r > 0$ be a real number smaller than the radius of convergence of F such that $\bar{B}(x_0,r) \subseteq U$ and $\tilde{F}(r) < 0$. Then f has exactly one simple zero ζ in $\bar{B}(x_0,r)$. In addition, the sequences $(x_k)_{k\in\mathbb{N}}$ recursively defined by $x_{k+1} := N(f;x_k)$ and $(t_k)_{k\in\mathbb{N}}$ recursively defined by $t_0 = 0$ and $t_{k+1} := N(\tilde{F};t_k)$ are well defined: x_k belongs to $\bar{B}(x_0,r)$, for all $k \geq 0$, converges quadratically to ζ, t_k is increasing and converges quadratically to the first nonnegative root r^- of $\tilde{F}$. Convergences compare this way:*

a. *$\|x_k - x_{k+1}\| \leq t_{k+1} - t_k$;*
b. *$\|Df(x_0)^{-1}f(x_k)\| \leq t_{k+1} - t_k$;*
c. *$\|x_k - \zeta\| \leq r^- - t_k$.*

Proof. Let us first deal with the degenerate case when $F'' = 0$. In this situation $x_k = \zeta$, $t_k = r^- = \beta$, for all $k \geq 1$ and the theorem trivially holds.

From now on, we assume $F'' \neq 0$. In particular this implies that $\tilde{F}$ is strictly convex on $[0,r]$. Since $\tilde{F}(0) \geq 0$ and $\tilde{F}(r) < 0$ we deduce that there exists a unique zero r^- of $\tilde{F}$ in $[0,r]$.

From convexity, we also deduce that t_k is increasing and converges to r^-. The dominating sequence technique consists in proving

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k. \tag{8}$$

For this purpose, we proceed by induction on k: if $k = 0$ then $\|x_1 - x_0\| = F(0) = \tilde{F}(0) = t_1$, since $\tilde{F}'(0) = -1$. Now, we assume that inequality (8) holds up to a certain index $k-1 \geq 0$. First, it is easy to check

$$\begin{aligned}\|x_k - x_0\| &\leq \|x_k - x_{k-1}\| + \cdots + \|x_1 - x_0\| \\ &\leq t_k - t_{k-1} + \cdots + t_1 - t_0 = t_k - t_0 \leq r^-. \end{aligned} \tag{9}$$

Since $\tilde{F}'$ does not vanish on $[0, r^-]$, Proposition 1.10 implies that the radius of convergence of $-1/\tilde{F}'(t)$ is larger than r^-. Therefore Propositions 1.14 and 1.4 yield:

$$\|Df(x_k)^{-1}Df(x_0)\| \leq -\frac{1}{\tilde{F}'(t_k)}. \tag{10}$$

On the other hand, by definition of x_k, one can write:

$$Df(x_0)^{-1}f(x_k) = Df(x_0)^{-1}(f(x_k) - f(x_{k-1}) - Df(x_{k-1})(x_k - x_{k-1})),$$

so that a second order Taylor formula with integral remainder gives:

$$Df(x_0)^{-1}f(x_k) = \int_0^1 (1-\tau)Df(x_0)^{-1}D^2f(x_{k-1} + \tau(x_k - x_{k-1}))(x_k - x_{k-1})^2 d\tau.$$

By Proposition 1.3, one has $\left[Df(x_0)^{-1}D^2f\right]_{x_0} \leq F''$ and then, using Proposition 1.4, we deduce: $\left\|Df(x_0)^{-1}f(x_k)\right\| \leq$

$$\int_0^1 (1-\tau)\tilde{F}''(\tau t_k + (1-\tau)t_{k-1})(t_k - t_{k-1})^2 d\tau = \tilde{F}(t_k). \tag{11}$$

Combining (10) and (11) leads to:

$$\begin{aligned} \|x_{k+1} - x_k\| &\leq \|Df(x_k)^{-1}f(x_k)\| \\ &\leq \|Df(x_k)^{-1}Df(x_0)\|\|Df(x_0)^{-1}f(x_k)\| \\ &\leq -\tilde{F}(t_k)/\tilde{F}'(t_k) = t_{k+1} - t_k, \end{aligned}$$

which gives (8). We deduce that the sequence $(x_k)_{k\in\mathbb{N}}$ converges to a limit that we write ζ. By construction, we have $f(\zeta) = 0$. From (9), we get $\zeta \in \bar{B}(x_0, r^-)$, thus ζ is a simple zero.

Next, part (b) follows from (11) this way:

$$\begin{aligned} \|Df(x_0)^{-1}f(x_k)\| \leq \tilde{F}(t_k) &= -\tilde{F}'(t_k)(t_{k+1} - t_k) \\ &\leq -\tilde{F}'(0)(t_{k+1} - t_k) = t_{k+1} - t_k. \end{aligned}$$

Lastly, part (c) comes from

$$\|x_k - \zeta\| \leq \sum_{i \geq k} \|x_i - x_{i+1}\| \leq \sum_{i \geq k} (t_{i+1} - t_i) = r^- - t_k.$$

It remains to show that ζ is the only zero of f in $\bar{B}(x_0, r)$. Let ζ' denote another zero in this ball. Let $R > r$ be smaller than the radius of convergence of F such that $\tilde{F}(R) \leq 0$. We introduce $\theta := \|x_0 - \zeta'\|/R < 1$ and claim

$$\|x_k - \zeta'\| \leq \theta^{2^k}(R - t_k), \text{ for all } k \geq 0, \tag{12}$$

from which immediately follows $\zeta = \zeta'$. We prove this claim by induction on k. The case $k = 0$ follows from the definition of θ. Assume that (12) holds at $k \geq 0$, by construction, one can write:

$$\begin{aligned} x_{k+1} - \zeta' &= -Df(x_k)^{-1}\left(f(x_k) + Df(x_k)(\zeta' - x_k)\right) \\ &= Df(x_k)^{-1}\int_0^1 (1-\tau)D^2f(x_k + \tau(\zeta' - x_k))(\zeta' - x_k)^2 d\tau, \end{aligned}$$

using again Taylor formula with integral remainder and $f(\zeta')=0$. Using the induction hypothesis and Proposition 1.14, we deduce:

$$\|x_{k+1}-\zeta'\| \leq -\frac{\theta^{2^{k+1}}}{\tilde{F}'(t_k)}\int_0^1 (1-\tau)\tilde{F}''(t_k+\tau\theta^{2^k}(R-t_k))(R-t_k)^2 d\tau.$$

Since $\tilde{F}''$ is increasing, we can omit the factor θ^{2^k-1} under the integral. Using $\tilde{F}(R)\leq 0$, we deduce:

$$\begin{aligned}\|x_{k+1}-\zeta'\| &\leq -\theta^{2^{k+1}}\frac{\tilde{F}(R)-\tilde{F}(t_k)-\tilde{F}'(t_k)(R-t_k)}{\tilde{F}'(t_k)}\\ &\leq \theta^{2^{k+1}}\frac{\tilde{F}(t_k)+\tilde{F}'(t_k)(R-t_k)}{\tilde{F}'(t_k)} = \theta^{2^{k+1}}(R-t_{k+1}). \qquad \square\end{aligned}$$

Specializing F to $F(f,x_0;.)$, we recover the following result by calculating an explicit expression for t_k:

Corollary 1.18. *(α-theorem) [55] If $\bar{B} := \bar{B}(x_0,(1+\sqrt{2}/2)\beta)\subseteq U$ and $\alpha < 3-2\sqrt{2}$ then f has exactly one simple zero ζ in $\bar{B}$ and the sequence $(x_k)_{k\in\mathbb{N}}$ recursively defined by $x_{k+1} := N(f;x_k)$ is well defined. In addition, x_k belongs to $\bar{B}$ and for all $k\geq 0$:*

a. $\|x_k - x_{k+1}\| \leq q^{2^k-1}\beta$,
b. $\|Df(x_0)^{-1}f(x_k)\| \leq q^{2^k-1}\beta$,
c. $\|x_k-\zeta\| \leq q^{2^k-1}r^-$,

where $q := \dfrac{2r^- - \beta}{2r^+ - \beta} < 1$ if $\alpha\neq 0$ and $q=0$ otherwise (in all cases we use the convention $q^0=1$).

Using (5), note that q rewrites in terms of α only:

$$q(\alpha) := \frac{4\alpha}{(1-\alpha+\sqrt{1-6\alpha+\alpha^2})^2}. \tag{13}$$

It follows that q and q/α are continuous on $[0, 3-2\sqrt{2})$.

Proof. The case $\alpha=0$ is straightforward with the convention. We now assume $\alpha\neq 0$. It is classical to get an explicit formula of t_k by means of introducing $s_k := \frac{t_k-r^-}{t_k-r^+}$. Then, writing $\tilde{F}(f,x_0;t) = 2\gamma\frac{(t-r^-)(t-r^+)}{1-\gamma t}$, we deduce

$$\frac{\tilde{F}'(f,x_0;t)}{\tilde{F}(f,x_0;t)} = \frac{1}{t-r^-}+\frac{1}{t-r^+}+\frac{\gamma}{1-\gamma t}$$

and

$$\begin{aligned} s_{k+1} &= \frac{t_k - r^- - \frac{\tilde{F}(f,x_0;t_k)}{\tilde{F}'(f,x_0;t_k)}}{t_k-r^+-\frac{\tilde{F}(f,x_0;t_k)}{\tilde{F}'(f,x_0;t_k)}} = \frac{(t_k-r^-)\frac{\tilde{F}'(f,x_0;t_k)}{\tilde{F}(f,x_0;t_k)}-1}{(t_k-r^+)\frac{\tilde{F}'(f,x_0;t_k)}{\tilde{F}(f,x_0;t_k)}-1}\\ &= s_k^2\frac{1+\gamma\frac{t_k-r^+}{1-\gamma t_k}}{1+\gamma\frac{t_k-r^-}{1-\gamma t_k}} = s_k^2 q_s,\end{aligned}$$

where $q_s := \frac{1-\gamma r^+}{1-\gamma r^-}$. From $s_0 = r^-/r^+$ and the definitions of r^- and r^+, it is easy to see that $\alpha < 3-2\sqrt{2}$ is equivalent to $q_s s_0 < 1$. Then, one can check $q = q_s s_0$ and an easy induction gives $s_k = q^{2^k-1}s_0$. It follows that s_k converges quadratically to 0 and t_k converges quadratically to r^-, since $t_k = r^- \frac{1-s_k/s_0}{1-s_k}$. In order to prove part (a) we write

$$t_{k+1} - t_k = \frac{r^+ - r^-}{1 - s_k} \frac{s_k - s_{k+1}}{1 - s_{k+1}}$$

and then deduce

$$\begin{aligned} \frac{t_{k+1} - t_k}{t_1 - t_0} &= \frac{s_k - s_{k+1}}{s_0 - s_1} \frac{1 - s_0}{1 - s_k} \frac{1 - s_1}{1 - s_{k+1}} \\ &= \frac{s_k}{s_0} \frac{1 - q_s s_k}{1 - q_s s_0} \frac{1 - s_0}{1 - s_k} \frac{1 - q_s s_0^2}{1 - q_s s_k^2} \leq \frac{s_k}{s_0}, \end{aligned}$$

the latter inequality follows from $q_s \leq 1$. Parts (a) and (b) follow from the previous theorem. Lastly, we deduce from part (c) of the previous theorem:

$$\|x_k - \zeta\| \leq r^- - t_k = r^- \frac{s_k}{s_0} \frac{1 - s_0}{1 - s_k} \leq r^- \frac{s_k}{s_0}. \qquad \square$$

1.6. **Local Compositional Inverse.** The first main ingredient of our location and approximation algorithms is the following quantitative version of the local compositional inverse function theorem:

Theorem 1.19. *Let $U \subseteq \mathbb{C}^n$ be an open subset, $f : U \to \mathbb{C}^n$ be an analytic map and $\zeta \in U$ be such that $Df(\zeta)$ is invertible. Let $\sigma \geq \|Df(\zeta)^{-1}\|$, $\gamma \geq \gamma(f;\zeta)$, $B_f := B\left(\zeta, \frac{1-\sqrt{2}/2}{\gamma}\right)$ and assume $B_f \subseteq U$. Then there exists a unique map g with the following properties:*

a. *g is defined and analytic in*
$$B_g := B\left(f(\zeta), \frac{1}{(3 + 2\sqrt{2})\sigma\gamma}\right);$$
b. *$g(B_g) \subseteq B_f$;*
c. *$f \circ g(b) = b$, for all $b \in B_g$;*
d. *For all $b \in B_g$ there exists only one $a \in B_f$ such that $f(a) = b$. In addition, we have $a = g(b)$ and $\|a - \zeta\| \leq (1 + \sqrt{2}/2)\beta(f - b;\zeta)$;*
e. *$[g - \zeta]_{f(\zeta)} \leq \dfrac{\sigma t}{1 - (3 + 2\sqrt{2})\sigma\gamma t}$.*

Proof. Let $b \in B_g$, $f_b(x) = f(x) - b$ defined on B_f. Then $\gamma(f_b;\zeta) = \gamma(f;\zeta) \leq \gamma$ and

$$\alpha(f_b;\zeta) \leq \gamma\sigma\|f(\zeta) - b\| < 3 - 2\sqrt{2}.$$

Hence Corollary 1.13 applies: $f_b(x)$ admits one simple zero a in B_f, we define $g(b) := a$. By construction, g is defined and analytic on B_g, hence parts (b), (c) and (d) hold. Lastly, part (e) follows from Corollary A.3(b) of Appendix A applied with with $\lambda = 1/\|Df(a)^{-1}\|$, $\rho = \gamma(f;a)$ and $\vartheta = 3 + 2\sqrt{2}$. $\square$

Remark that, according to the notation of this proof, Corollary 1.18 implies that Newton's operator on f_b converges quadratically from ζ to a. Parts (a) to (d) of this theorem and their proof are taken from to [4, Chapter 8, Theorem 7]. Part (e) is also a direct consequence of line -3 of the proof of [8, Corollary 3.4]. The above proof of part (e) follows from stronger results on the behavior of majorant series under local compositional inversion, that are given in Appendix A, and that generalize those of Dedieu *et al.* given in [8, Section 3].

2. Reduction to One Variable

This section contains the last ingredients that are used in the next sections to reduce the location and approximation problem from several variables to one variable, as explained in the introduction. We start with recalling the main result we shall use on location of clusters of analytic functions from [12, Section 1]. Then, as required by the univariate approximation algorithm of [12, Section 4], we provide a function $\mathcal{B}_{m,l}$ that computes approximations of $\beta_{m,l}$. Next, we prove formula (2) from the introduction, that relates $D_y^l h$ to $\mathsf{g}^{[l]}$ and, lastly, we provide bounds on translation with respect to z of point estimates of maps $h(z,.)$.

2.1. Clusters of Zeros of Univariate Functions.

For the sake of completeness we recall the following result.

Theorem 2.1. [12, Corollary 1.8] *Let f denote an analytic function defined on a connected open subset $U \subseteq \mathbb{C}$, let $m \geq 1$ be an integer, $l \in \{0, \ldots, m-1\}$, $z \in U$ be such that $f^{(m)}(z) \neq 0$,*

$$\frac{m-l}{m}\frac{m+1}{m+1-l}\alpha_{m,l}(f;z) \leq 1/9$$

and $\bar{B}(z, 3\frac{m-l}{m}\beta_{m,l}(f;z)) \subseteq U$. Then, $f^{(l)}$ has $m-l$ zeros, counting multiplicities, in $\bar{B}(z, 3\frac{m-l}{m}\beta_{m,l}(f;z))$ and $\bar{B}(z, \frac{m+1-l}{3(m+1)\gamma_m(f;z)}) \cap U$.

Let us recall from [12, Section 2] that if $\alpha_{m,l}(f;z)$ is sufficiently small and if z is in the convex hull of the cluster of zeros of $f^{(l)}$ located by the previous proposition then the diameter of this cluster is about $\beta_{m,l}(f;z)$. Roughly speaking, this means that one can confound the diameter of this cluster and $\beta_{m,l}(f;z)$, for any point in the convex hull of the cluster. This is why we focus on such quantities $\beta_{m,l}(f;z)$ to approximate clusters in Section 4.

From [12, Section 4.1], we recall bounds on translation of $\alpha_{m,l}, \beta_{m,l}$ and γ_m estimates that will be particularly useful in practice in our last section.

Proposition 2.2. [12, Proposition 4.3] *Assume that U is connected, let $\zeta \in U$, $m \geq 1$ be an integer such that $f^{(m)}(\zeta) \neq 0$ and $l \in \{0, \ldots, m-1\}$. Let $\gamma_m := \gamma_m(f;\zeta)$ and $\beta_{m,l} := \beta_{m,l}(f;\zeta)$, for short. Let $z \in U$, $r := |z-\zeta|$ be such that $u := \gamma_m(f;\zeta)r < 1-(1/2)^{1/(m+1)}$, then $f^{(m)}(z) \neq 0$ and*

a. $\alpha_{m,l}(f;z) \leq \frac{1}{\psi_m(u)^2}\left(\alpha_{m,l}(1-u)^{\frac{l+1}{m-l}} + (2m-1)u\right)$;

b. $\beta_{m,l}(f;z) \leq \frac{1-u}{\psi_m(u)}\left(\beta_{m,l}(1-u)^{\frac{l+1}{m-l}} + (2m-1)r\right)$;

c. $\gamma_m(f,z) \leq \frac{\gamma_m}{\psi_m(u)(1-u)}$;

d. $\left|\frac{f^{(m)}(\zeta)}{f^{(m)}(z)}\right| \leq \frac{(1-u)^{m+1}}{\psi_m(u)}$;

e. $\left|\dfrac{f^{(m)}(z)}{f^{(m)}(\zeta)}\right| \le \dfrac{1}{(1-u)^{m+1}}$.

2.2. **Approximation of $\beta_{m,l}$.** Let U be a connected open subset of $\mathbb{C}$, in this subsection f denotes an analytic function defined on U. Let x_0 and x_1 be two points in U, $m \ge 1$ be an integer such that $f^m(x_0) \ne 0$ and let $l \in \{0, \dots, m-1\}$. We show that $\beta_{m,l}(f; x_1)$ can be approximated from the sole knowledge of a truncated Taylor expansion of f at x_0 and upper bounds on $\gamma_m(f; .)$ at x_0 and x_1. We introduce the following functions, that will be used in Section 4:

$$\mathcal{B}_{m,l}(f, x_0; x_1) := \beta_{m,l}(p; x_1), \quad v_{m,l} := \min\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{m+1}}, \frac{1}{2m-l}\right),$$

$$\tau_{m,l,0}(v) := \frac{(2m-l)^2(1-v)}{\psi_m(v)}, \quad \tau_{m,l,1}(v) := 1 + \frac{(2m-l)v}{\psi_m(v)},$$

where p denotes the unique polynomial of degree at most $2m - l - 1$ such that $f(x) - p(x) \in \mathcal{O}_{x_0}((x-x_0)^{2m-l})$. These objects satisfy the requirements stated in [12, Section 4.2], namely:

Proposition 2.3. *Let $r := |x_0 - x_1|$, $\bar{\gamma}_m \ge \max(\gamma_m(f; x_0), \gamma_m(f; x_1))$, $v = \bar{\gamma}_m r$, if $v < v_{m,l}$ then*

$$\mathcal{B}_{m,l}(f, x_0; x_1) \le \tau_{m,l,1}(v)\beta_{m,l}(f; x_1) + \tau_{m,l,0}(v)\bar{\gamma}_m r^2,$$
$$\beta_{m,l}(f; x_1) \le \tau_{m,l,1}(v)\mathcal{B}_{m,l}(f, x_0; x_1) + \tau_{m,l,0}(v)\bar{\gamma}_m r^2.$$

Proof. It suffices to set $i = 2m - l$ in the next lemma. □

Lemma 2.4. *Let $i \ge m+1$ be an integer and $p(y)$ denote the unique polynomial of degree at most $i-1$ such that $f(x) - p(x) \in \mathcal{O}_{x_0}((x-x_0)^i)$. Let $r := |x_1 - x_0|$, $v := \gamma_m(f; x_0)r$ and assume $v < 1 - (1/2)^{1/(m+1)}$ and $iv < 1$, then $|\beta_{m,l}(f; x_1) - \beta_{m,l}(p; x_1)| \le$*

$$\frac{(iv)^{\frac{i-m}{m-l}}}{\psi_m(v)}\Big(\min(\beta_{m,l}(f; x_1), \beta_{m,l}(p; x_1)) + (1-v)ir\Big).$$

Proof. Let $\gamma_m := \gamma_m(f; x_0)$ and $\sigma_m := m!/|f^{(m)}(x_0)| = m!/|p^{(m)}(x_0)|$. We introduce the majorant series

$$R := \frac{\gamma_m^{i-m} t^i}{1 - \gamma_m t},$$

so that $\sigma_m [f-p]_{x_0} \le R$. Let $l \le k \le m$. By [12, Lemma 4.1], one has

$$\frac{R^{(k)}}{k!} \le \frac{\gamma_m^{i-m}\binom{i}{k} t^{i-k}}{(1-\gamma_m t)^{k+1}} \le \frac{(i\gamma_m t)^{i-m}(it)^{m-k}}{(1-\gamma_m t)^{k+1}}.$$

Observe that $\gamma_m(p; x_0) \le \gamma_m$ and therefore both $f^{(m)}(x_1)$ and $p^{(m)}(x_1)$ do not vanish, according to Proposition 2.2, and:

$$\left|\frac{m!}{\sigma_m p^{(m)}(x_1)}\right| \le \frac{(1-v)^{m+1}}{\psi_m(v)}, \qquad \left|\frac{m!}{\sigma_m f^{(m)}(x_1)}\right| \le \frac{(1-v)^{m+1}}{\psi_m(v)}.$$

Then we start with:

$$\left|\frac{m!f^{(k)}(x_1)}{k!f^{(m)}(x_1)} - \frac{m!p^{(k)}(x_1)}{k!p^{(m)}(x_1)}\right| \le \left|\frac{m!f^{(k)}(x_1)}{k!p^{(m)}(x_1)} - \frac{m!p^{(k)}(x_1)}{k!p^{(m)}(x_1)}\right| + \left|\frac{m!f^{(k)}(x_1)}{k!p^{(m)}(x_1)} - \frac{m!f^{(k)}(x_1)}{k!f^{(m)}(x_1)}\right|. \tag{14}$$

Using majorant series evaluation via Proposition 1.4, we bound the first term of the right-hand side of the latter inequality:

$$\begin{aligned}\left|\frac{m!f^{(k)}(x_1)}{k!p^{(m)}(x_1)} - \frac{m!p^{(k)}(x_1)}{k!p^{(m)}(x_1)}\right| &\le \left|\frac{m!}{p^{(m)}(x_1)}\right|\left|\frac{f^{(k)}(x_1)}{k!} - \frac{p^{(k)}(x_1)}{k!}\right| \\ &\le \left|\frac{m!}{\sigma_m p^{(m)}(x_1)}\right|\left|\frac{R^{(k)}(r)}{k!}\right| \\ &\le \frac{(1-v)^{m+1}}{\psi_m(v)}\frac{(iv)^{i-m}(ir)^{m-k}}{(1-v)^{k+1}} \\ &\le \frac{(1-v)^{m-k}}{\psi_m(v)}(iv)^{i-m}(ir)^{m-k}.\end{aligned}$$

As for the second term we get:

$$\begin{aligned}\left|\frac{m!f^{(k)}(x_1)}{k!p^{(m)}(x_1)} - \frac{m!f^{(k)}(x_1)}{k!f^{(m)}(x_1)}\right| &\le \left|\frac{m!f^{(k)}(x_1)}{k!f^{(m)}(x_1)}\right|\left|\frac{m!}{p^{(m)}(x_1)}\right|\left|\frac{f^{(m)}(x_1)}{m!} - \frac{p^{(m)}(x_1)}{m!}\right| \\ &\le \beta_{m,l}(f;x_1)^{m-k}\left|\frac{m!}{\sigma_m p^{(m)}(x_1)}\right|\left|\frac{R^{(m)}(r)}{m!}\right| \\ &\le \beta_{m,l}(f;x_1)^{m-k}\frac{(1-v)^{m+1}}{\psi_m(v)}\frac{(iv)^{i-m}}{(1-v)^{m+1}} \\ &\le \beta_{m,l}(f;x_1)^{m-k}\frac{(iv)^{i-m}}{\psi_m(v)}.\end{aligned}$$

Then using the assumption $iv < 1$, we conclude:

$$|\beta_{m,l}(f;x_1) - \beta_{m,l}(p;x_1)| \le \frac{(iv)^{\frac{i-m}{m-l}}}{\psi_m(v)}\left(\beta_{m,l}(f;x_1) + (1-v)ir\right),$$

which is the first half of the claimed inequality. The second half is obtained in a similar way, starting from (14) and exchanging the roles of f and p. □

2.3. **Deflated Maps.** For any integer $l \ge 0$, in this text, we study the lth *deflated map* $(\mathtt{f}, \mathtt{g}^{[l]})$ defined in (1) from $f = (\mathtt{f}, \mathtt{g})$, according to the following recursive formal definition:

$$\mathtt{g}^{[0]} = \mathtt{g}, \quad \mathtt{g}^{[l+1]} = \frac{\det(D(\mathtt{f}, \mathtt{g}^{[l]}))}{\det(D_x\mathtt{f})}, \text{ for } l \ge 0,$$

where det denotes the determinant map, the implicit basis being considered is (x, y). For any (x_0, y_0) such that $D_x\mathtt{f}(x_0, y_0)$ is invertible, the relation with $h(f, x_0, y_0; z, y)$ is as follows.

Lemma 2.5. *Let $l \geq 0$, $(x_0, y_0) \in U$ be such that $D_x\mathtt{f}(x_0, y_0)$ is invertible and $z_0 := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x_0, y_0)$. The following relation holds in a neighborhood of (z_0, y_0):*

$$D_y^l h(f, x_0, y_0; z, y) = \mathtt{g}^{[l]} \circ \Phi(\mathtt{f}, x_0, y_0; z, y).$$

Proof. For short, we let

$$h(z, y) := h(f, x_0, y_0; z, y), \;\; \Phi(z, y) := \Phi(\mathtt{f}, x_0, y_0; z, y).$$

Let X denote the first $n-1$ coordinates of $D_y\Phi(z, y)$ and Y the last one. By construction, we have $Y = 1$ and $D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f} \circ \Phi(z, y) = z$. In a neighborhood of (z_0, y_0), differentiating this equality with respect to y yields

$$D_x\mathtt{f}(\Phi(z, y))X + D_y\mathtt{f}(\Phi(z, y)) = 0.$$

On the other hand we have $D_yh(z, y) = D\mathtt{g}(\Phi(z, y))D_y\Phi(z, y)$, hence:

$$\begin{aligned} D_yh(z, y) &= D_x\mathtt{g}(\Phi(z, y))X + D_y\mathtt{g}(\Phi(z, y)) \\ &= -D_x\mathtt{g}(\Phi(z, y))D_x\mathtt{f}(\Phi(z, y))^{-1}D_y\mathtt{f}(\Phi(z, y)) + D_y\mathtt{g}(\Phi(z, y)). \end{aligned}$$

For $l = 1$, the conclusion is a consequence of the classical Schur complement formula, that comes from mapping determinant on each factor of

$$\begin{pmatrix} \mathrm{Id} & 0 \\ D_x\mathtt{g}(\Phi(z, y)) & -1 \end{pmatrix} \begin{pmatrix} D_x\mathtt{f}(\Phi(z, y))^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_x\mathtt{f}(\Phi(z, y)) & D_y\mathtt{f}(\Phi(z, y)) \\ D_x\mathtt{g}(\Phi(z, y)) & D_y\mathtt{g}(\Phi(z, y)) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Id} & D_x\mathtt{f}(\Phi(z, y))^{-1}D_y\mathtt{f}(\Phi(z, y)) \\ 0 & -D_yh(z, y) \end{pmatrix}.$$

An easy induction on l concludes the proof. □

From a practical point of view, and by means of classical computations with power series, starting from (x_0, y_0), one can obtain truncated Taylor series expansions of Φ at (z_0, y_0) in order to deduce such expansions for h via this lemma.

2.4. **z-Translation of Point Estimates.** Here we focus on quantifying the variations of $\beta_{m,l}(h(z, .); y)$ and $\gamma_m(h(z, .); y)$ when z varies, for fixed y, where h denotes an analytic map from an open subset U of $\mathbb{C}^{n-1} \times \mathbb{C}$ and with values in $\mathbb{C}$.

Proposition 2.6. *Let $h(z, y)$ be an analytic map defined on a connected open neighborhood U of (z_0, y_0) to $\mathbb{C}$. Let $\lambda \geq 0$ and $\rho \geq 0$ be two real numbers satisfying $[h - h(z_0, y_0)]_{(z_0,y_0)} \leq \frac{\lambda t}{1-\rho t}$. Let $m \geq 1$ and $l \in \{0, \ldots, m-1\}$ be integers such that $|D_y^m h(z_0, y_0)| \neq 0$ and let $\sigma_m \geq m!/|D_y^m h(z_0, y_0)|$. Let z_1 be such that $(z_1, y_0) \in U$ and*

$$\mu := \frac{\rho}{1 - \rho\|z_1 - z_0\|}, \qquad e := \left((m+1)\sigma_m \frac{\lambda\|z_1 - z_0\|}{1 - \rho\|z_1 - z_0\|}\right)^{1/m}.$$

If $\rho\|z_1 - z_0\| < 1$ and $\mu e < 1$ then $|D_y^m h(z_1, y_0)| \neq 0$ and

a. $m!/|D_y^m h(z_1, y_0)| \leq \dfrac{m!/|D_y^m h(z_0, y_0)|}{1 - (\mu e)^m}$ *and*
$m!/|D_y^m h(z_0, y_0)| \leq (1 + (\mu e)^m)m!/|D_y^m h(z_1, y_0)|$;

b. $\beta_{m,l}(h(z_1,.);y_0) \leq \dfrac{\beta_{m,l}(h(z_0,.);y_0) + \frac{m}{m+1}\mu^{\frac{l}{m-l}}e^{\frac{m}{m-l}}}{1-(\mu e)^m}$ *and*

$$\beta_{m,l}(h(z_0,.);y_0) \leq (1+(\mu e)^m)\beta_{m,l}(h(z_1,.);y_0) + \frac{m}{m+1}\mu^{\frac{l}{m-l}}e^{\frac{m}{m-l}};$$

c. $\gamma_m(h(z_1,.);y_0) \leq \dfrac{\gamma_m(h(z_0,.);y_0) + \frac{m+2}{m+1}\mu}{1-(\mu e)^m}$ *and*

$$\gamma_m(h(z_0,.);y_0) \leq (1+(\mu e)^m)\gamma_m(h(z_1,.);y_0) + \frac{m+2}{m+1}\mu.$$

Proof. Let $r := \|z_1 - z_0\|$. If $\rho = 0$ then $m = 1$ and the proposition holds trivially. Now we assume $\rho > 0$. Inequalities $\|D_z^j D_y^k h(z_0,y_0)\| \leq \|D^{j+k}h(z_0,y_0)\|$, for all $j \geq 0$ and $k \geq 0$, rewrite into $\left[D_y^k h(.,y_0)\right]_{z_0} \leq \left[D^k h\right]_{(z_0,y_0)}$. Differentiating majorant series, thanks to Proposition 1.3, we obtain, for any $k \geq 1$:

$$\left[\frac{D_y^k h(.,y_0)}{k!} - \frac{D_y^k h(z_0,y_0)}{k!}\right]_{z_0} \leq \left[\frac{D^k h}{k!} - \frac{D^k h(z_0,y_0)}{k!}\right]_{(z_0,y_0)} \leq \frac{\lambda\rho^{k-1}}{(1-\rho t)^{k+1}} - \lambda\rho^{k-1}.$$

Evaluating at $t = r$, using Proposition 1.4, we deduce:

$$\left|\frac{|D_y^k h(z_1,y_0)|}{k!} - \frac{|D_y^k h(z_0,y_0)|}{k!}\right| \leq \frac{\lambda\rho^{k-1}}{(1-\rho r)^{k+1}}(1-(1-\rho r)^{k+1}) \leq \frac{(k+1)\lambda\rho^k r}{(1-\rho r)^{k+1}} = (k+1)\mu^k\frac{\lambda r}{1-\rho r},$$

since $1-(1-\rho r)^{k+1}$ is an increasing concave function of r. Observe that the latter inequality also holds for $k = 0$. Rewriting the previous expressions in terms of e and μ, we find:

$$\left|\frac{m!|D_y^k h(z_1,y_0)|}{k!|D_y^m h(z_0,y_0)|} - \frac{m!|D_y^k h(z_0,y_0)|}{k!|D_y^m h(z_0,y_0)|}\right| \leq \frac{k+1}{m+1}\mu^{k-m}(\mu e)^m. \tag{15}$$

Letting $k := m$ in this inequality we obtain:

$$1-(\mu e)^m \leq |D_y^m h(z_1,y_0)|/|D_y^m h(z_0,y_0)| \leq 1+(\mu e)^m, \tag{16}$$

which yields part (a). Combining (15) and (16), we deduce, for any $k \geq 0$:

$$\frac{m!|D_y^k h(z_1,y_0)|}{k!|D_y^m h(z_1,y_0)|} \leq \frac{\frac{m!|D_y^k h(z_0,y_0)|}{k!|D_y^m h(z_0,y_0)|} + \frac{k+1}{m+1}\mu^{k-m}(\mu e)^m}{1-(\mu e)^m} \tag{17}$$

and

$$\frac{m!|D_y^k h(z_0,y_0)|}{k!|D_y^m h(z_0,y_0)|} \leq (1+(\mu e)^m)\frac{m!|D_y^k h(z_1,y_0)|}{k!|D_y^m h(z_1,y_0)|} + \frac{k+1}{m+1}\mu^{k-m}(\mu e)^m.$$

For short, we let $\beta_{m,l} := \beta_{m,l}(h(z_0,.);y_0)$ and $\gamma_m := \gamma_m(h(z_0,.);y_0)$. For part (b), let us consider $l \leq k \leq m-1$, then (16) and (17) respectively lead to:

$$\beta_{m,l}(h(z_1,.);y_0) \leq \frac{\beta_{m,l} + \mu^{-1}\sup_{l\leq k\leq m-1} b_{m,k}(\mu e)^{\frac{m}{m-k}}}{1-(\mu e)^m}$$

and

$$\beta_{m,l} \leq (1+(\mu e)^m)\beta_{m,l}(h(z_1,.);y_0) + \mu^{-1}\sup_{l\leq k\leq m-1} b_{m,k}(\mu e)^{\frac{m}{m-k}},$$

where $b_{m,k} := \left(\frac{k+1}{m+1}\right)^{\frac{1}{m-k}}$. Part (b) follows from $\sup_{l\leq k\leq m-1}(\mu e)^{\frac{m}{m-k}} = (\mu e)^{\frac{m}{m-l}}$ and

$$\sup_{l\leq k\leq m-1} b_{m,k} = \frac{m}{m+1}.$$

This equality can be seen as a consequence of the concavity of the log function:

$$\begin{aligned}\log(b_{m,k}) &= \frac{\log(k+1)-\log(m+1)}{(m+1)-(k+1)}\\ &\leq \frac{\log(m)-\log(m+1)}{(m+1)-m} = \log(b_{m,m-1}),\end{aligned}$$

which is an equality for $k = m-1$. As for part (c), calculations are very similar. Letting $k \geq m+1$, and using $\mu e < 1$, inequalities (16) and (17) respectively lead to:

$$\gamma_m(h(z_1,.);y_0) \leq \frac{\gamma_m + \mu\sup_{k\geq m+1} c_{m,k}}{1-(\mu e)^m}$$

and

$$\gamma_m \leq (1+(\mu e)^m)\gamma_m(h(z_1,.);y_0) + \mu\sup_{k\geq m+1} c_{m,k},$$

where $c_{m,k} := \left(\frac{k+1}{m+1}\right)^{\frac{1}{k-m}}$. Part (c) follows from

$$\sup_{k\geq m+1} c_{m,k} = \frac{m+2}{m+1}.$$

Again, this equality can be seen as a consequence of the concavity of the log function:

$$\begin{aligned}\log(c_{m,k}) &= \frac{\log(k+1)-\log(m+1)}{(k+1)-(m+1)}\\ &\leq \frac{\log(m+2)-\log(m+1)}{(m+2)-(m+1)} = \log(c_{m,m+1}).\end{aligned}$$ □

3. Cluster Location

In this section, we carry on with the notation of the introduction: $\mathtt{f} : U_{\mathtt{f}} \to \mathbb{C}^{n-1}$ and $\mathtt{g} : U_{\mathtt{g}} \to \mathbb{C}$ are analytic maps defined on maximal analyticity domains. We present a method for locating clusters of zeros of the lth deflated map $(\mathtt{f}, \mathtt{g}^{[l]})$.

In order to perform this location around a given point (x_0, y_0), we focus on locating zeros of $D_y^l h(\mathtt{f}, \mathtt{g}, x_0, y_0; 0, .)$ around y_0, which reduces to a univariate situation. Unfortunately, it is not possible to compute point

$$\lambda_\Phi := \sigma_x; \quad \rho_\Phi := (3+2\sqrt{2})\sigma_x\gamma_x;$$
$$\lambda := \lambda_{\mathtt{g}}\lambda_\Phi; \quad \rho := \rho_\Phi + \lambda_\Phi\rho_{\mathtt{g}};$$
$$\bar\lambda := \frac{\lambda}{(1-\rho(\beta_x+\|z_0'\|))^2}; \quad \bar\rho := \frac{\rho}{1-\rho(\beta_x+\|z_0'\|)};$$
$$\bar\mu := \frac{\bar\rho}{1-\bar\rho\|z_0'\|}; \quad \bar e := \left((m+1)\sigma_m\frac{\bar\lambda\|z_0'\|}{1-\bar\rho\|z_0'\|}\right)^{1/m};$$
$$\bar\beta_{m,l} := \frac{\beta_{m,l}+\frac{m}{m+1}\bar\mu^{\frac{l}{m-l}}\bar e^{\frac{m}{m-l}}}{1-(\bar\mu\bar e)^m}; \quad \bar\gamma_m := \frac{\gamma_m+\frac{m+2}{m+1}\bar\mu}{1-(\bar\mu\bar e)^m};$$
$$r_y^- := 3\frac{m-l}{m}\bar\beta_{m,l}; \quad r_y^+ := \frac{m+1-l}{3(m+1)\bar\gamma_m};$$
$$r_x^- := \frac{\lambda_\Phi(\beta_x+r_y^-)}{1-\rho_\Phi(\beta_x+r_y^-)}.$$

TABLE 1. Auxiliary quantities for Theorem 3.1

estimates of h at $(0, y_0)$ in general. We have to content ourselves with estimates at (z_0', y_0), where $z_0' := D_x\mathtt{f}(x_0,y_0)^{-1}\mathtt{f}(x_0',y_0)$ is small enough for x_0' is obtained from x_0 by means of Newton's iteration. Our algorithm together with its main properties are presented in the following theorem.

Theorem 3.1. *Let $\mathtt{f} : U_{\mathtt{f}} \to \mathbb{C}^{n-1}$ and $\mathtt{g} : U_{\mathtt{g}} \to \mathbb{C}$ be maximal analytic maps with $U_{\mathtt{f}} \cap U_{\mathtt{g}} \neq \emptyset$. Let $(x_0, y_0) \in U_{\mathtt{f}} \cap U_{\mathtt{g}}$ be such that $D_x\mathtt{f}(x_0, y_0)$ is invertible. Let $\Sigma(x, y) := \Sigma(\mathtt{f}, x_0, y_0; x, y)$, $h(z, y) := h(\mathtt{f}, \mathtt{g}, x_0, y_0; z, y)$. Let $m \geq 1$, $l \in \{0, \ldots, m-1\}$, κ be the first integer such that $2^\kappa \geq m - l$, $x_0' := N^\kappa(\mathtt{f}(., y_0); x_0)$ and $z_0' := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x_0', y_0)$.*

Let $\lambda_{\mathtt{g}}$, $\rho_{\mathtt{g}}$, β_x, γ_x, σ_x, $\beta_{m,l}$, γ_m, σ_m be given nonnegative real numbers. Let the auxiliary quantities be as defined in Table 1, and let us assume that all the conditions in Table 2 hold.

Then, with r_x^-, r_x^+, and r_y^- as defined in Table 2, $(\mathtt{f}, \mathtt{g}^{[l]})$ has $m - l$ zeros, counting multiplicities, in

$$B_\Sigma \cap B_{\mathtt{g}} \cap (\bar B(x_0, r_x^-) \times \bar B(y_0, r_y^-))$$

but also in

$$B_\Sigma \cap B_{\mathtt{g}} \cap (\mathbb{C}^{n-1} \times \bar B(y_0, r_y^+)),$$

where

$$B_\Sigma := B\left((x_0, y_0), \frac{1-\sqrt{2}/2}{\gamma_x}\right), \quad B_{\mathtt{g}} := B\left((x_0, y_0), \frac{1}{\rho_{\mathtt{g}}}\right).$$

Before entering the proof, let us explain the main idea and feature of the method. First observe that if $(x_0, y_0) = \zeta$ is a multiple zero of multiplicity m then the theorem applies. By continuity, it follows that the process actually locates clusters of embedding dimension 1. Informally speaking, by construction, we shall see below in (20) that z_0' belongs to $\mathcal{O}(\beta_x^{m-l})$,

(L_1) $[\mathtt{g}-\mathtt{g}(x_0,y_0)]_{(x_0,y_0)} \leq \dfrac{\lambda_{\mathtt{g}} t}{1-\rho_{\mathtt{g}} t}$;
(L_2) $\sigma_x \geq \|D\Sigma(x_0,y_0)^{-1}\|$;
(L_3) $\beta_x \geq \beta(\Sigma-(0,y_0);x_0,y_0)$; $\quad \gamma_x \geq \gamma(\Sigma;x_0,y_0)$;
(L_4) $\rho(\beta_x+\|z_0'\|) < 1$;
(L_5) $\lvert D_y^m h(z_0',y_0)\rvert \neq 0$; $\quad \sigma_m \geq m!/\lvert D_y^m h(z_0',y_0)\rvert$;
(L_6) $\beta_{m,l} \geq \beta_{m,l}(h(z_0',.);y_0)$; $\quad \gamma_m \geq \gamma_m(h(z_0',.);y_0)$;
(L_7) $\bar\rho\|z_0'\| < 1$; $\quad \bar\mu\bar e < 1$;
(L_8) $\dfrac{m-l}{m}\dfrac{m+1}{m+1-l}\bar\beta_{m,l}\bar\gamma_m \leq 1/9$;

TABLE 2. Assumptions for Theorem 3.1

hence $\bar e^{\frac{m}{m-l}} \in \mathcal{O}(\beta_x)$. Therefore, we have $\bar\beta_{m,l} \in \mathcal{O}(\beta_{m,l}+\beta_x)$, which is the motivation for the definition of κ above.

Proof of Theorem 3.1. Recall that Φ denotes the local inverse of Σ and, for short, we let

$$z_0 := D_x\mathtt{f}(x_0,y_0)^{-1}\mathtt{f}(x_0,y_0),\ \Phi(z,y) := \Phi(\mathtt{f},x_0,y_0;z,y),$$

$\phi(y)$ denotes the $n-1$ first coordinates of $\Phi(0,y)$ and we introduce:

$$B_\Phi := B\left((z_0,y_0),\frac{3-2\sqrt{2}}{\sigma_x\gamma_x}\right),\ B_h := B\left((z_0,y_0),\frac{1}{\rho}\right)$$

$$\text{and } B_\phi := B\left(y_0,\frac{1}{\rho}-\beta_x\right).$$

From (L_3) (resp. (L_1)), Σ (resp. $\mathtt{g}$) is well defined on B_Σ (resp. $B_{\mathtt{g}}$). Using (L_3) and (L_2), Theorem 1.19 ensures that Φ is well defined on B_Φ, $\Phi(B_\Phi) \subseteq B_\Sigma$ and

$$[\Phi-(x_0,y_0)]_{(z_0,y_0)} \leq \frac{\lambda_\Phi t}{1-\rho_\Phi t}. \tag{18}$$

Composing this series majoration with that of (L_1), Corollary 1.9 applied with $h = \mathtt{g}\circ\Phi$ gives $[h-h(z_0,y_0)]_{(z_0,y_0)} \leq \lambda t/(1-\rho t)$ and $\Phi(B_h) \subseteq B_\Sigma\cap B_{\mathtt{g}}$. From their definitions, observe that $B_h \subseteq B_\Phi$. It follows that h is well defined on B_h.

From the definition of z_0, one has $\|z_0\| = \beta(\Sigma-(0,y_0);x_0,y_0) \leq \beta_x$ (from (L_3)), hence (L_4) implies $(0,y_0) \in B_h$. We deduce that $\phi(y)$ is well defined on B_ϕ and part (d) of Theorem 1.19 implies

$$\{(x,y)\in B_\Sigma \mid y\in B_\phi,\ \mathtt{f}(x,y)=0\} = \{(\phi(y),y) \mid y \in B_\phi\},$$

which, combined to Lemma 2.5, leads to:

$$\begin{aligned}\Big\{(x,y)\in B_\Sigma\cap B_{\mathtt{g}} \mid y\in B_\phi,\ \mathtt{f}(x,y)=\mathtt{g}^{[l]}(x,y)=0\Big\}\\ =\Big\{(\phi(y),y)\mid y\in B_\phi,\ D_y^l h(0,y)=0\Big\}.\end{aligned} \tag{19}$$

From (L_2) and the definition of Σ, we have $\sigma_x \geq 1$ and therefore

$$\gamma_x \leq \sigma_x\gamma_x = (3-2\sqrt{2})\rho_\Phi \leq (3-2\sqrt{2})\rho.$$

Successively using (L_3) and (L_4) we deduce:

$$\beta(\Sigma-(0,y_0);x_0,y_0)\gamma(\Sigma;x_0,y_0) \leq \alpha_x := \beta_x\gamma_x \leq (3-2\sqrt{2})\rho\beta_x < 3-2\sqrt{2}.$$

hence Corollary 1.18 gives:

$$\|z_0'\| < q(\alpha_x)^{2^\kappa-1}\beta_x,$$

where the function q is defined in (13). Since $q(\alpha_x)<1$ and by the definition of κ, we deduce

$$\|z_0'\| < q(\alpha_x)^{m-l-1}\beta_x \leq \beta_x. \tag{20}$$

Although we referred to this inequality just before the proof, as a motivation of the definition of κ, we will not use it in the remainder of the proof.

Using the inequalities $\|z_0'-z_0\| \leq \beta_x + \|z_0'\|$ and (L_4), Corollary 1.6 on majorant series translation leads to:

$$\big[h-h(z_0',y_0)\big]_{(z_0',y_0)} \leq \frac{\bar\lambda t}{1-\bar\rho t}. \tag{21}$$

We are now ready to deduce point estimates of $h(0,.)$ from $h(z_0',.)$ at y_0. Using (L_5), (L_6) and (L_7), Proposition 2.6 yields: $D_y^m h(0,y_0)\neq 0$ and

$$\beta_{m,l}(h(0,.);y_0) \leq \bar\beta_{m,l}, \quad \gamma_m(h(0,.);y_0)\leq\bar\gamma_m.$$

Using (L_8), we now proceed to zero location, via Theorem 2.1: $D_y^l h(0,y)$ admits $m-l$ zeros Z_h in $\bar B(y_0,r_y^-)$ and in $\bar B(y_0,r_y^+)$. Remark that $\bar\gamma_m \geq \bar\mu \geq \bar\rho \geq \rho/(1-\beta_x\rho)$, from which follows:

$$\beta_x + r_y^- \leq \beta_x + r_y^+ < \beta_x + \frac{1}{\bar\gamma_m} \leq \frac{1}{\rho}. \tag{22}$$

We deduce $Z_h\subseteq \bar B(y_0,r_y^-)\subseteq \bar B(y_0,r_y^+)\subseteq B_\phi$. From (19), we deduce, for any $r_y\in\{r_y^-,r_y^+\}$:

$$\begin{aligned}&\Big\{(x,y)\in B_\Sigma\cap B_{\mathtt{g}} \mid y\in \bar B(y_0,r_y),\ \mathtt{f}(x,y)=\mathtt{g}^{[l]}(x,y)=0\Big\}\\ &=\Big\{(\phi(y),y)\mid y\in \bar B(y_0,r_y),\ D_y^l h(0,y)=0\Big\}\\ &=\{(\phi(y),y)\mid y\in Z_h\}.\end{aligned}$$

For $r_y=r_y^+$, this gives the second half of the conclusion. As for the first half, from (22) and the evaluation of (18), via Proposition 1.4, for any $\zeta_y\in Z_h$ we have

$$\|\Phi(0,\zeta_y)-\Phi(z_0,y_0)\| \leq r_x^-,$$

which concludes the proof. □

4. Cluster Approximation

In this section, we present an approximation algorithm for clusters of embedding dimension 1, with the same features as the one given in [12] for univariate functions: either quadratic convergence holds or the current iterate lies at a distance of the cluster which is about its diameter. More generally, our operator depends on two parameters $l \in \{0, \ldots, m-1\}$ and $l' \leq l$, in order to approximate clusters of $(\mathtt{f}, \mathtt{g}^{[l']})$ by using the univariate Schröder operator on $D_y^l h(f, x_0, y_0; z, .)$.

We carry on with the notation of the introduction. We recall that $\mathtt{f} : U_{\mathtt{f}} \to \mathbb{C}^{n-1}$ and $\mathtt{g} : U_{\mathtt{g}} \to \mathbb{C}$ are analytic maps defined on maximal analyticity domains. We assume that $U := U_{\mathtt{f}} \cap U_{\mathtt{g}}$ is not empty, and, for short, we let $h(z, y) := h(\mathtt{f}, \mathtt{g}, x_0, y_0; x, y)$, $\Sigma(x, y) := \Sigma(\mathtt{f}, x_0, y_0; x, y)$, $\Phi(z, y) := \Phi(\mathtt{f}, x_0, y_0; z, y)$, and $(\phi(y), y) := \Phi(0, y)$, where $(x_0, y_0) \in U$ denotes the initial point of the iteration. The functions $\mathcal{B}_{m,l'}$, $\tau_{m,l,0}$ and $\tau_{m,l,1}$ are the ones introduced in Section 2.2.

We use the following quantities, that come from the univariate situation [12]:

$$\theta_{m,l,\delta} := \delta\frac{1}{m} + \frac{m+1}{(m-l+1)(m-l)},$$

$$u_{m,l,\delta} := \max\left(u \geq 0 \mid u < 1 - (1/2)^{1/(l+2)} \text{ and } \frac{\theta_{m,l,\delta} u}{\psi_{l+1}(u)} \leq 1\right),$$

$$C_{m,l,l',\delta}(u) := \frac{1-u}{\psi_m(u)} \frac{(1-u)^{\frac{l'+1}{m-l'}} + \frac{\theta_{m,l,\delta}(2m-1)}{\psi_{l+1}(u)}}{\left(1 - \frac{\theta_{m,l,\delta} u}{\psi_{l+1}(u)}\right)^2}.$$

4.1. Algorithm.

The approximation algorithm depends on the initial point (x_0, y_0) and on three positive real numbers r_y, $\mathcal{G}_y$ and $\mathcal{G}_z$ that will be assigned later.

An iteration of the algorithm computes (x_{k+1}, y_{k+1}) from (x_k, y_k). A rough description is as follows: first, the Newton iteration on $\mathtt{f}(., y_k)$ is applied a certain number of times with starting point x_k to compute a new value x'_k; then we compute z'_k as the $n-1$ first coordinates of $\Sigma(x'_k, y_k)$ and the Schröder operator is applied on $D_y^l h(z_k, .)$ with starting point y_k, this gives a value y'_k; then a discussion takes place to determine which of y_k and y'_k should be taken for y_{k+1}; finally, y_{k+1} is used in one Newton iteration on $\mathtt{f}(., y_{k+1})$ with starting point x_k to compute x_{k+1}.

More formally, we introduce the operator $N_{m,l,l'}(x, y)$, defined by the following algorithm, in which κ represents the smallest integer such that $2^\kappa \geq 2(m - l')$. We also introduce the flag $\mathcal{F}^y_{m,l,l'}(x_k, y_k)$ with values in the set of symbols $\{\infty, +, -, 1\}$, that keeps track of the branchings.

$(x_{k+1}, y_{k+1}) := N_{m,l,l'}(x_k, y_k)$ is defined by

(1) $x'_k := N^\kappa(\mathtt{f}(., y_k); x_k)$;
(2) $z'_k := D_x\mathtt{f}(x_0, y_0)^{-1}\mathtt{f}(x'_k, y_k)$;
(3) if $D_y^{l+1} h(z'_k, y_k) = 0$
(4) then

(5) $\quad y_{k+1} := y_k; \qquad \mathcal{F}^y_{m,l,l'}(x_k, y_k) := \infty;$
(6) else
(7) $\quad y'_k := N_{m-l}(D^l_y h(z'_k, .); y_k);$
(8) $\quad$ if $y'_k \notin \bar{B}(y_k, 2r_y)$
(9) $\quad$ then
(10) $\quad\quad y_{k+1} := y_k; \qquad \mathcal{F}^y_{m,l,l'}(x_k, y_k) := \infty;$
(11) $\quad$ else
(12) $\quad\quad$ if $\mathcal{B}_{m,l'}(h(z'_k, .), y_k; y'_k) > \mathcal{G}_y |y_k - y'_k|^2$
(13) $\quad\quad$ then
(14) $\quad\quad\quad$ if $\mathcal{B}_{m,l'}(h(z'_k, .), y_k; y'_k) < \beta_{m,l'}(h(z'_k, .); y_k)$
(15) $\quad\quad\quad$ then
(16) $\quad\quad\quad\quad y_{k+1} := y'_k; \qquad \mathcal{F}^y_{m,l,l'}(x_k, y_k) := +;$
(17) $\quad\quad\quad$ else
(18) $\quad\quad\quad\quad y_{k+1} := y_k; \qquad \mathcal{F}^y_{m,l,l'}(x_k, y_k) := -;$
(19) $\quad\quad$ else
(20) $\quad\quad\quad y_{k+1} := y'_k; \qquad \mathcal{F}^y_{m,l,l'}(x_k, y_k) := 1;$
(21) $x_{k+1} := N(\mathtt{f}(., y_{k+1}); x_k);$

In a similar way, we introduce $\mathcal{F}^z_{m,l,l'}(x_k, y_k)$ that takes the value 1 if

$$\mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1}) \leq \mathcal{G}_z \|z'_k\|^{1/(m-l')}$$

and 0 otherwise. These flags are to be used for stopping the iteration.

Informally speaking, x'_k is obtained from Newton's iteration in order to get $\|z'_k\|^{1/(m-l')}$ of the second order, that is in $\mathcal{O}(\|(x_k, y_k) - \zeta\|^2)$. Then we apply Schröder's operator to $D^l_y h(z'_k, .)$ at y_k. This is where we deeply use [12, Section 4] and different cases happen. Step (10) corresponds to y'_k getting far from y_k, which implies that y_k is close to the cluster of $D^l_y h(z'_k, .)$. On the other hand, step (20) corresponds to the fact that y'_k is closer to this cluster at the second order. In the meanwhile, the test of step (14) determines the one among y_k and y'_k which is closer to this cluster. At the end, the correction between y_{k+1} and y_k is propagated to the x coordinates at step (21).

The convergence analysis of the sequence $(x_k, y_k)_{k\in\mathbb{N}}$ is presented in the following theorem. The way of turning it into a

Theorem 4.1. *Let $\zeta := (\zeta_x, \zeta_y) \in U$ be such that $\mathtt{f}(\zeta) = 0$ and $D_x\mathtt{f}(\zeta)$ is invertible. Let $(x_0, y_0) \in \bar{B}(\zeta, r)$, for a given $r \geq 0$. Let m be such that $D^m_y h(0, \zeta_y) \neq 0$, $l \in \{0, \ldots, m-1\}$ and $l' \leq l$. Let r_y, γ_x, σ_x, $\lambda_{\mathtt{g}}$, $\rho_{\mathtt{g}}$, $\bar{\gamma}_x$, $\bar{\delta}_x$, $\bar{\beta}_{m,l'}$, $\bar{\gamma}_m$, $\bar{\sigma}_m$ be given real numbers. Let the auxiliary quantities be as defined in Table 3, and let us assume that all the conditions in Table 4 hold.*

Then, there exists a cluster $\mathcal{Z}_{0,\zeta_y}$ of $m - l'$ zeros of $D^{l'}_y h(0, .)$ in $\bar{B}(\zeta_y, r_y)$, counting multiplicities. We further assume that ζ_y belongs to the convex hull of $\mathcal{Z}_{0,\zeta_y}$. With $\mathcal{G}_y$ and $\mathcal{G}_z$ as defined in Table 3, the operator $N_{m,l,l'}$ is well defined, so that we can consider the sequence $(x_k, y_k)_{k\geq 0}$ formally defined by induction according to:

$$(x_{k+1}, y_{k+1}) := N_{m,l,l'}(x_k, y_k),$$

$$\delta := 1; \quad \lambda_\Phi := \sigma_x; \quad \rho_\Phi := (3+2\sqrt{2})\sigma_x\gamma_x;$$

$$\lambda := \lambda_{\mathtt{g}}\lambda_\Phi; \quad \rho := \rho_\Phi + \lambda_\Phi\rho_{\mathtt{g}};$$

$$l_\phi := \frac{\lambda_\Phi}{1-\rho_\Phi r}; \quad L_\phi := 1 + l_\phi; \quad r_x := 2L_\phi r; \quad u_x := \bar{\gamma}_x r_x; \quad \bar{r} := 4r_y;$$

$$L_z := L_\phi^2 \left(\frac{\bar{\gamma}_x}{\psi(u_x)}\right)^{2-1/(m-l')} \left(\frac{\bar{\delta}_x}{1-u_x}\right)^{1/(m-l')}; \quad r_z := (L_z r^2)^{m-l'};$$

$$\bar{\lambda} := \frac{\lambda}{(1-\rho\bar{r})^2}; \quad \bar{\rho} := \frac{\rho}{1-\rho\bar{r}};$$

$$\bar{\mu} := \frac{\bar{\rho}}{1-\bar{\rho}r_z}; \quad L_{\bar{e}} := \left((m+1)\left(\frac{m}{m+1}\right)^{m-l'} \bar{\mu}^{l'}\bar{\sigma}_m \frac{\bar{\lambda}}{1-\bar{\rho}r_z}\right)^{1/m};$$

$$\bar{e} := \left((m+1)\bar{\sigma}_m \frac{\bar{\lambda}r_z}{1-\bar{\rho}r_z}\right)^{1/m}; \quad u := \bar{\gamma}_m r_y; \quad v := 2u; \quad \bar{u} := 3u;$$

$$\mathcal{C} := \tau_{m,l',1}(v)C_{m,l,l',\delta}(u) + \tau_{m,l',0}(v); \quad \mathcal{G}_y := \mathcal{C}\bar{\gamma}_m;$$

$$\bar{\mathcal{C}} := \tau_{m,l',1}(v)\mathcal{C} + \tau_{m,l',0}(v);$$

$$\underline{\kappa} := \frac{\tau_{m,l',1}(v)}{1-\frac{\tau_{m,l',0}(v)}{\mathcal{C}}}; \quad \bar{\kappa} := \tau_{m,l',1}(v) + \frac{\tau_{m,l',0}(v)}{\mathcal{C}};$$

$$\chi := \frac{1-\bar{u}}{\psi_m(\bar{u})}\left((1-\bar{u})^{\frac{l'+1}{m-l'}} + \frac{2m-1}{u_{m,l,\delta}}\right); \quad \Xi := \underline{\kappa}\chi;$$

$$T_1 := \frac{1}{1-(\bar{\mu}\bar{e})^m}\frac{1-u}{\psi_m(u)}\left((1-u)^{\frac{l'+1}{m-l'}} + \frac{3^{\frac{m-l'}{m}}(2m-1)}{1-(\bar{\mu}\bar{e})^m}\right);$$

$$T_2 := \frac{L_{\bar{e}}^{m/(m-l')}}{1-(\bar{\mu}\bar{e})^m}\left(1 + \frac{1}{1-(\bar{\mu}\bar{e})^m}\frac{3^{\frac{m-l'}{m}}(2m-1)(1-u)}{\psi_m(u)}\right);$$

$$T_\beta := 2\Xi T_1; \quad \mathcal{G}_z := 2\Xi T_2;$$

$$T_y := 3\frac{m-l'}{m}\left((1+(\bar{\mu}\bar{e})^m)\bar{\kappa} + L_{\bar{e}}^{m/(m-l')}/\mathcal{G}_z\right);$$

$$L_{y,1} := 3\frac{m-l'}{m}\Big[(1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}\bar{\gamma}_m\left(1+3\frac{m-l'}{m}L_{\bar{e}}^{m/(m-l')}L_z r\right)^2 \left(1-3\frac{m-l'}{m}(1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}v\right)^{-2} + L_{\bar{e}}^{m/(m-l')}L_z\Big];$$

$$L_{y,2} := 3\frac{m-l'}{m}\left((1+(\bar{\mu}\bar{e})^m)\bar{\kappa}\mathcal{G}_z + L_{\bar{e}}^{m/(m-l')}\right)L_z;$$

$$L_y := \max(L_{y,1}, L_{y,2}); \quad L_x := \frac{4\bar{\gamma}_x}{\psi(u_x)}L_\phi^2 + L_\phi L_y.$$

TABLE 3. Auxiliary quantities for Theorem 4.1

(A_1)	$\gamma_x \geq \gamma(\Sigma;\zeta)$; $\sigma_x \geq \|D\Sigma(\zeta)^{-1}\|$;
(A_2)	$[\mathrm{g}-\mathrm{g}(\zeta)]_\zeta \leq \dfrac{\lambda_{\mathrm{g}} t}{1-\rho_{\mathrm{g}} t}$;
(A_3)	$\rho_\phi r < 1$;
(A_4)	$\bar{\gamma}_x \geq \max(\gamma(\Sigma;\phi(y),y) \mid y \in \bar{B}(\zeta_y, r))$;
(A_5)	$\bar{\delta}_x \geq \max(\|D\Sigma(\phi(y),y)\| \mid y \in \bar{B}(\zeta_y, r))$;
(A_6)	$u_x < \frac{5-\sqrt{17}}{4}$;
(A_7)	$r_y \geq r$;
(A_8)	$r_y \geq 3\dfrac{m-l'}{m}\max(\beta_{m,l'}(h(z,.);y) \mid z \in \bar{B}(0,r_z),\ y \in \bar{B}(\zeta_y, r))$;
(A_9)	$\rho(r_z+\bar{r}) < 1$;
(A_{10})	$+\infty > \bar{\sigma}_m \geq \max(m!/\|D_y^m h(0,y)\| \mid y \in \bar{B}(\zeta_y,\bar{r}))$;
(A_{11})	$\bar{\beta}_{m,l'} \geq \max(\beta_{m,l'}(h(z,.);y) \mid z \in \bar{B}(0,r_z),\ y \in \bar{B}(\zeta_y,\bar{r}))$;
(A_{12})	$\bar{\gamma}_m \geq \max(\gamma_m(h(z,.);y) \mid z \in \bar{B}(0,r_z),\ y \in \bar{B}(\zeta_y,\bar{r}))$;
(A_{13})	$\bar{\rho} r_z < 1$; $\quad \bar{\mu}\bar{e} < 1$;
(A_{14})	$\dfrac{m-l'}{m}\dfrac{m+1}{m+1-l'}\bar{\beta}_{m,l'}\bar{\gamma}_m \leq 1/9$;
(A_{15})	$r_y\dfrac{m+1}{m+1-l'}\bar{\gamma}_m \leq 1/12$;
(A_{16})	$u < u_{m,l,\delta}$; $\quad \bar{u} < 1-(1/2)^{1/(m+1)}$; $\quad v < v_{m,l'}$;
(A_{17})	$3\dfrac{m-l'}{m}(1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}} v < 1$;
(A_{18})	$L_y r < 1$;
(A_{19})	$L_x r < 1$.

TABLE 4. Assumptions for Theorem 4.1

and we can define K as the first integer such that

$$\mathcal{F}^y_{m,l,l'}(x_K,y_K) \neq 1 \text{ and } \mathcal{F}^z_{m,l,l'}(x_K,y_K) = 0,$$

or $+\infty$ if no such integer exists.

Then, for all $k \leq K$, (x_k,y_k) is well defined, and with L_x, T_y and T_β, as defined in Table 3, the sequence $(x_k,y_k)_{k\geq 0}$ converges to the cluster $\mathcal{Z}_{0,\zeta_y}$ as follows:

a. *For all $k < K$ we have:*

$$\|(x_{k+1},y_{k+1})-\zeta\| \leq L_x\|(x_k,y_k)-\zeta\|^2 \leq r.$$

b. *If K is finite then y_{K+1} is well defined and satisfies:*

$$|y_{K+1}-\zeta_y| \leq T_y\mathcal{B}_{m,l'}(h(z'_K,.),y_K;y_{K+1}) \tag{23}$$

and

$$\mathcal{B}_{m,l'}(h(z'_K,.), y_K; y_{K+1}) \leq T_\beta \beta_{m,l'}(h(0,.);\zeta_y). \tag{24}$$

Subsections 4.2 to 4.8 are devoted to the proof of this theorem. The notation is the same as in the theorem, and, for short, we write $\beta_{m,l'} := \beta_{m,l'}(h(0,.);\zeta_y)$. We proceed by induction on k: we assume that $((x_j, y_j))_{0\leq j\leq k}$ is well defined up to a certain index $k \leq K$ and that all its elements belong to $\bar{B}(\zeta, r)$. The proof of part (b) is addressed at the end, namely in subsection 4.8.

4.2. **Definition Domains.** We first provide definition domains for all the maps involved in the algorithm. We introduce:

$$B_\Sigma := B\left(\zeta, \frac{1-\sqrt{2}/2}{\gamma_x}\right), \; B_{\mathtt{g}} := B\left(\zeta, \frac{1}{\rho_{\mathtt{g}}}\right),$$

$$B_\Phi := B\left((0,\zeta_y), \frac{1}{\rho_\Phi}\right), \; B_h := B\left((0,\zeta_y), \frac{1}{\rho}\right).$$

Lemma 4.2. *Σ (resp. Φ) is well defined on B_Σ (resp. B_Φ). h is well defined as the composition $\mathtt{g}\circ\Phi$ on B_h.*

Proof. Using (A_1) and (A_2) we have $B_\Sigma \subseteq U_{\mathtt{f}}$, $B_{\mathtt{g}} \subseteq U_{\mathtt{g}}$ and Theorem 1.19 about local inversion ensures that Φ is well defined on B_Φ, $\Phi(B_\Phi) \subseteq B_\Sigma$ and

$$[\Phi - \zeta]_{(0,\zeta_y)} \leq \frac{\lambda_\Phi t}{1-\rho_\Phi t}. \tag{25}$$

Then, according to Corollary 1.9 on majorant series composition applied to $h = \mathtt{g}\circ\Phi$, we get

$$[h - h(0,\zeta_y)]_{(0,\zeta_y)} \leq \frac{\lambda t}{1-\rho t} \tag{26}$$

and $\Phi(B_h) \subseteq B_{\mathtt{g}}$, which means that $h = \mathtt{g}\circ\Phi$ is well defined on B_h. □

4.3. **Uniform Convergence to the Curve.** According to our hypotheses, $\mathtt{f}(x,y) = 0$ defines a smooth curve in a neighborhood of ζ. We perform a uniform convergence analysis to this curve for the operator used in step (1) of the algorithm. We start with two lemmas.

Lemma 4.3. *For all $(a,b) \in \bar{B}(\zeta_x, r) \times \bar{B}(\zeta_y, r)$ we have*

$$\|(\phi(b), b) - \zeta\| \leq l_\phi |b - \zeta_y|,$$
$$\|a - \phi(b)\| \leq L_\phi \|(a,b) - \zeta\|. \tag{27}$$

Proof. Using (A_3), the evaluation of (25) by means of Proposition 1.4 yields:

$$\|(\phi(b), b) - \zeta\| = \|\Phi(0,b) - \Phi(0,\zeta_y)\| \leq l_\phi |b - \zeta_y|,$$

which implies

$$\|a - \phi(b)\| \leq \|a - \zeta_x\| + \|\phi(b) - \zeta_x\| \leq L_\phi \|(a,b) - \zeta\|. \qquad \square$$

Lemma 4.4. *For any $(a,b) \in \bar{B}(\zeta_x, r) \times \bar{B}(\zeta_y, r)$, and any integers $j \geq 0$ and $p \leq 2^j$, $a' := N^j(\mathtt{f}(.,b), a)$ is well defined and*

$$\|a' - \phi(b)\| \leq \left(\frac{\bar{\gamma}_x \|a - \phi(b)\|}{\psi(u_x)}\right)^{p-1} \|a - \phi(b)\| \leq r_x.$$

Proof. By the previous lemma, $\|a - \phi(b)\| \leq 2L_\phi r = r_x$ holds. Then, using (A_4), we deduce $\gamma(\Sigma; \phi(b), b)\|a - \phi(b)\| \leq u_x$, which, via ($A_6$), implies $\frac{u_x}{\psi(u_x)} < 1$. Thus Theorem 1.16 gives:

$$\|a' - \phi(b)\| \leq \left(\frac{\bar{\gamma}_x \|a - \phi(b)\|}{\psi(u_x)}\right)^{2^j - 1} \|a - \phi(b)\| \leq r_x.$$

Using $\frac{u_x}{\psi(u_x)} < 1$ yields the claimed bound. □

We are now able to deduce that $z'_k \in \bar{B}(0, r_z)$, which will be used several times in the remainder of the proof, without explicit reference:

Corollary 4.5. $\|z'_k\| \leq L_z^{m-l'} \|(x_k, y_k) - \zeta\|^{2(m-l')} \leq r_z$.

Proof. We apply the previous lemma to $(a, b) := (x_k, y_k)$, $j := \kappa$ and $p := 2(m - l')$:

$$\|x'_k - \phi(y_k)\| \leq \left(\frac{\bar{\gamma}_x \|x_k - \phi(y_k)\|}{\psi(u_x)}\right)^{2(m-l')-1} \|x_k - \phi(y_k)\|.$$

By means of (27) (instantiated at $(a, b) := (x_k, y_k)$), we deduce:

$$\|x'_k - \phi(y_k)\| \leq \left(\frac{\bar{\gamma}_x}{\psi(u_x)}\right)^{2(m-l')-1} L_\phi^{2(m-l')} \|(x_k, y_k) - \zeta\|^{2(m-l')}.$$

Using (A_4) and (A_5), we deduce the following series majoration:

$$[\Sigma - (0, y_k)]_{\Phi(0, y_k)} \leq \frac{\bar{\delta}_x t}{1 - \bar{\gamma}_x t},$$

which evaluates at (x'_k, y_k) since $\bar{\gamma}_x \|x'_k - \phi(y_k)\| \leq u_x < 1$ (from (A_6)), by means of Proposition 1.4:

$$\begin{aligned}\|z'_k\| &= \|\Sigma(x'_k, y_k) - (0, y_k)\| \\ &\leq \frac{\bar{\delta}_x \|x'_k - \phi(y_k)\|}{1 - u_x} \leq L_z^{m-l'} \|(x_k, y_k) - \zeta\|^{2(m-l')} \leq r_z.\end{aligned}$$ □

4.4. **Uniform z-Translation.** From the previous result, $\|z'_k\|^{1/(m-l')}$ belongs to $\mathcal{O}(\|(x_k, y_k) - \zeta\|^2)$. We are now ready to apply our bound of Section 2.4 on z-translation in a uniform way with respect to y. We start with a uniform series majoration of h. First of all, it is important to notice that (A_9) implies:

$$B_h \supseteq \bar{B}(0, r_z) \times \bar{B}(\zeta_y, \bar{r}).$$

Lemma 4.6. *For all $b \in \bar{B}(\zeta_y, \bar{r})$, we have:* $[h - h(0, b)]_{(0,b)} \leq \dfrac{\bar{\lambda} t}{1 - \bar{\rho} t}$.

Proof. This directly follows from Corollary 1.6, using (A_9) and (26). □

Lemma 4.7. *For all $(c, b) \in \bar{B}(0, r_z) \times \bar{B}(\zeta_y, \bar{r})$, we have $D_y^m h(c, b) \neq 0$ and*

$$\beta_{m,l'}(h(0, .); b) \leq (1 + (\bar{\mu}\bar{e})^m)\beta_{m,l'}(h(c, .); b) + L_{\bar{e}}^{m/(m-l')} \|c\|^{1/(m-l')},$$

$$\beta_{m,l'}(h(c, .); b) \leq \frac{\beta_{m,l'}(h(0, .); b) + L_{\bar{e}}^{m/(m-l')} \|c\|^{1/(m-l')}}{1 - (\bar{\mu}\bar{e})^m}.$$

Proof. Using (A_{10}), (A_{13}) and Lemma 4.6, Proposition 2.6 (with $(z_0, y_0) := (0, b)$ and $z_1 := c$) implies these bounds. □

4.5. **Uniform Cluster Location.** Now, we show quantitative results about clusters of zeros of $D_y^{l'}h(c,.)$ when c varies.

Lemma 4.8. *For any $(c,b) \in \bar{B}(0,r_z) \times \bar{B}(\zeta_y,\bar{r})$, there exists a cluster $\mathcal{Z}_{c,b}$ of $m-l'$ zeros of the analytic extension of $D_y^{l'}h(c,.)$ in*

$$\bar{B}\left(b, 3\frac{m-l'}{m}\beta_{m,l'}(h(c,.);b)\right) \text{ and } \bar{B}\left(b, \frac{m+1-l'}{3(m+1)\gamma_m(h(c,.);b)}\right).$$

In addition, if $b \in \bar{B}(\zeta_y, r)$ then

$$\bar{B}\left(b, 3\frac{m-l'}{m}\beta_{m,l'}(h(c,.);b)\right) \subseteq \bar{B}(b, r_y).$$

Proof. The location of $\mathcal{Z}_{c,b}$ directly follows from (A_{11}), (A_{12}) and (A_{14}) and Theorem 2.1. The latter ball inclusion rephrases (A_8). □

Lemma 4.9. *For any $(c,b) \in \bar{B}(0,r_z) \times \bar{B}(\zeta_y, r)$ and any $b' \in \bar{B}(b, 3r_y)$, we have $\mathcal{Z}_{c,b} = \mathcal{Z}_{c,b'}$. In particular, for any b'' in the convex hull of $\mathcal{Z}_{c,b}$, we have:*

$$|b''-b'| \le 3\frac{m-l'}{m}\beta_{m,l'}(h(c,.);b').$$

Proof. From (A_7), observe that $b' \in \bar{B}(\zeta_y,\bar{r})$, thus applying the previous lemma at (c,b') gives the existence of a cluster $\mathcal{Z}_{c,b'}$ contained in the ball $\bar{B}(b', 3\frac{m-l'}{m}\beta_{m,l'}(h(c,.);b'))$ and also in $\bar{B}(b', \frac{m+1-l'}{3(m+1)\gamma_m(h(c,.);b')})$. Then, for any $b'' \in \mathcal{Z}_{c,b}$ one has:

$$\begin{aligned}
|b''-b'| &\le |b''-b| + |b-b'| \\
&\le r_y + |b-b'| \quad \text{(using the previous lemma)} \\
&\le 4r_y \\
&\le \frac{m+1-l'}{3(m+1)\bar{\gamma}_m} \quad \text{(using } (A_{15})\text{)} \\
&\le \frac{m+1-l'}{3(m+1)\gamma_m(h(c,.);b')} \quad \text{(using } (A_{12})\text{)}.
\end{aligned}$$

We deduce $\mathcal{Z}_{c,b} = \mathcal{Z}_{c,b'}$. □

We now specialize these lemmas to our situation. For this purpose, we first notice:

Lemma 4.10. $|y_{k+1} - \zeta_y| \le 3r_y \le \bar{r}$.

Proof. By construction, we have $|y_k - y_{k+1}| \le 2r_y$. Since (A_7) implies $|y_k - \zeta_y| \le r \le r_y$, we deduce: $|y_{k+1}-\zeta_y| \le |y_k - y_{k+1}| + |y_k - \zeta_y| \le 3r_y \le \bar{r}$. □

This statement will be invoked several times in the remainder of the proof without explicit reference. Lemma 4.9 is only used to prove the next two corollaries. The first statement of Theorem 4.1 about the location of the cluster $\mathcal{Z}_{0,\zeta_y}$ around ζ_y is a consequence of Lemma 4.8. From now, we assume that ζ_y belongs to the convex hull of this cluster.

Corollary 4.11. *There exist $m - l'$ zeros of $D_y^{l'} h(0,.)$ in $\bar{B}(\zeta_y, r_y)$ and*

$$|y_{k+1} - \zeta_y| \leq 3\frac{m-l'}{m}\beta_{m,l'}(h(0,.); y_{k+1}).$$

Proof. The inequality directly follows from Lemma 4.9, applied with $(c, b) := (0, \zeta_y)$, $b' := y_{k+1}$ and $b'' := \zeta_y$ (thanks to Lemma 4.10). ☐

The last corollary concerns the location of the cluster around (z'_k, y_k).

Corollary 4.12. *There exist $m - l'$ zeros of $D_y^{l'} h(z'_k, .)$ in $\bar{B}(y_k, r_y)$. We denote by ζ_k a point lying in the convex hull of this cluster. Then one has:*

$$|\zeta_k - \zeta_y| \leq 3\frac{m-l'}{m}\beta_{m,l'}(h(z'_k,.); \zeta_y) \leq r_y. \tag{28}$$

Proof. The first part directly follows from Lemma 4.8 applied with $(c, b) := (z'_k, y_k)$. From (A_7), one has $|y_k - \zeta_y| \leq r \leq r_y$, hence the first inequality of (28) follows from applying Lemma 4.9 with $(c, b) := (z'_k, y_k)$, $b' := \zeta_y$ and $b'' := \zeta_k$. The second inequality directly follows from (A_8). ☐

4.6. **Uniform Cluster Approximation.** Let ζ_k be the one defined in Corollary 4.12. Here we show that the approximation algorithm of [12, Section 4.4] applies to $D_y^l h(z'_k, .)$.

Lemma 4.13. *The following alternative holds about y_{k+1}:*

a. *If $\mathcal{F}^y_{m,l,l'}(x_k, y_k) \neq 1$ then*

$$\beta_{m,l'}(h(z'_k,.); y_{k+1}) \leq \bar{\kappa}\mathcal{B}_{m,l'}(h(z'_k,.), y_k; y_{k+1}), \tag{29}$$

$$\mathcal{B}_{m,l'}(h(z'_k,.), y_k; y_{k+1}) \leq \Xi\beta_{m,l'}(h(z'_k,.); \zeta_k); \tag{30}$$

b. *If $\mathcal{F}^y_{m,l,l'}(x_k, y_k) = 1$ then*

$$\beta_{m,l'}(h(z'_k,.); y_{k+1}) \leq \bar{\mathcal{C}}\bar{\gamma}_m|y_k - y_{k+1}|^2.$$

Proof. It suffices to check that the conditions of [12, Theorem 4.5] are satisfied with the analytic extension of $h(z'_k, .)$ at ζ_k with r_y. Namely, making use of $|\zeta_k - \zeta_y| \leq r_y$ (Corollary 4.12), we check:

- The function $\mathcal{B}_{m,l'}$ satisfies the required properties, thanks to Proposition 2.3;
- $D_y^m h(z'_k, \zeta_k) \neq 0$ by Lemma 4.7;
- $\bar{\gamma}_m \geq \gamma_m(h(z'_k,.); \zeta_k)$, thanks to (A_{12});
- $\bar{\gamma}_m \geq \max(\gamma_m(h(z'_k,.); y) \mid y \in \bar{B}(\zeta_k, 3r_y))$, thanks to (A_{12}) again;
- $u < u_{m,l,\delta}$, $\bar{u} < 1 - (1/2)^{1/(m+1)}$, $v < v_{m,l'}$, from (A_{16}).

Part (b) is directly extracted from [12, Theorem 4.5, part (c)]. As for part (a), we distinguish two cases.

First, if $D_y^{l+1} h(z'_k, y_k) = 0$ or $D_y^{l+1} h(z'_k, y_k) \neq 0$ and $y'_k \notin \bar{B}(y_k, 2r_y)$ then from [12, Theorem 4.5, part (a)] one has:

$$\beta_{m,l'}(h(z'_k,.); y_k) \leq \chi\beta_{m,l'}(h(z'_k,.); \zeta_k).$$

In the second case, $y'_k \in \bar{B}(y_k, 2r_y)$ but $\mathcal{B}_{m,l'}(h(z'_k,.), y_k; y'_k) > \mathcal{G}_y|y_k - y'_k|^2$, from [12, Theorem 4.5, part (b)] one has

$$\min(\beta_{m,l'}(h(z'_k,.); y_k), \beta_{m,l'}(h(z'_k,.); y'_k)) \leq \chi\beta_{m,l'}(h(z'_k,.); \zeta_k) \tag{31}$$

and

$$\beta_{m,l'}(h(z'_k,.);y'_k) \le \bar{\kappa}\mathcal{B}_{m,l'}(h(z'_k,.),y_k;y'_k),$$
$$\mathcal{B}_{m,l'}(h(z'_k,.),y_k;y'_k) \le \underline{\kappa}\beta_{m,l'}(h(z'_k,.);y'_k).$$

Since $\bar{\kappa} \ge 1$ and $\beta_{m,l'}(h(z'_k,.);y_k) = \mathcal{B}_{m,l'}(h(z'_k,.),y_k;y_k)$, we deduce (29). From (31) and $\underline{\kappa} \ge 1$, we also obtain

$$\min(\beta_{m,l'}(h(z'_k,.);y_k), \mathcal{B}_{m,l'}(h(z'_k,.),y_k;y'_k)) \le \underline{\kappa}\chi\beta_{m,l'}(h(z'_k,.);\zeta_k),$$

which yields (30) by definition of y_{k+1}. □

4.7. **Proof of Part (a) of Theorem 4.1.** We distinguish two cases:

Case 1: In this case, we assume $\mathcal{F}^y_{m,l,l'}(x_k,y_k) = 1$. Combining part (b) of Lemma 4.13 and Lemma 4.7 (applied with $(c,b) := (z'_k, y_{k+1})$), we deduce:

$$\beta_{m,l'}(h(0,.);y_{k+1}) \le (1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}\bar{\gamma}_m|y_{k+1}-y_k|^2 + L_{\bar{e}}^{m/(m-l')}\|z'_k\|^{1/(m-l')}$$

hence Corollaries 4.5 and 4.11 imply

$$|y_{k+1}-\zeta_y| \le 3\frac{m-l'}{m}\Big((1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}\bar{\gamma}_m|y_{k+1}-y_k|^2 + L_{\bar{e}}^{m/(m-l')}L_z\|(x_k,y_k)-\zeta\|^2\Big). \tag{32}$$

Then, from $|y_k - y_{k+1}| \le |y_k-\zeta_y| + |y_{k+1}-\zeta_y|$ we deduce:

$$|y_k-y_{k+1}| \le |y_k-\zeta_y| + 3\frac{m-l'}{m}\Big((1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}v|y_{k+1}-y_k| + L_{\bar{e}}^{m/(m-l')}L_z\|(x_k,y_k)-\zeta\|^2\Big).$$

Using (A_{17}) yields

$$\begin{aligned}|y_k-y_{k+1}| &\le \frac{|y_k-\zeta_y| + 3\frac{m-l'}{m}L_{\bar{e}}^{m/(m-l')}L_z\|(x_k,y_k)-\zeta\|^2}{1-3\frac{m-l'}{m}(1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}v}\\ &\le \frac{1+3\frac{m-l'}{m}L_{\bar{e}}^{m/(m-l')}L_z r}{1-3\frac{m-l'}{m}(1+(\bar{\mu}\bar{e})^m)\bar{\mathcal{C}}v}\|(x_k,y_k)-\zeta\|.\end{aligned} \tag{33}$$

Combining (32) and (33) leads to

$$|y_{k+1}-\zeta_y| \le L_{y,1}\|(x_k,y_k)-\zeta\|^2. \tag{34}$$

Case 2: Now we examine the case when

$$\mathcal{F}^y_{m,l,l'}(x_k,y_k) \ne 1 \text{ but } \mathcal{F}^z_{m,l,l'}(x_k,y_k) = 1.$$

Successively using Lemma 4.7 (with $(c,b) := (z'_k, y_{k+1})$) and part (a) of Lemma 4.13, we deduce

$$\beta_{m,l'}(h(0,.);y_{k+1}) \le \Big((1+(\bar{\mu}\bar{e})^m)\bar{\kappa}\mathcal{G}_z + L_{\bar{e}}^{m/(m-l')}\Big)\|z'_k\|^{1(m-l')},$$

which leads to, by means of Corollaries 4.11 and 4.5:

$$|y_{k+1}-\zeta_y| \le L_{y,2}\|(x_k,y_k)-\zeta\|^2. \tag{35}$$

This concludes this second case.

We are now ready to conclude the proof of part (a) of Theorem 4.1. According to the definition of L_y and (A_{18}), bounds (34) and (35) imply:

$$|y_{k+1} - \zeta_y| \leq L_y \|(x_k, y_k) - \zeta\|^2 \leq \|(x_k, y_k) - \zeta\| \leq r. \tag{36}$$

From Lemma 4.4 (applied with $(a, b) := (x_k, y_{k+1})$), we have:

$$\|x_{k+1} - \phi(y_{k+1})\| \leq \frac{\bar{\gamma}_x}{\psi(u_x)} \|x_k - \phi(y_{k+1})\|^2.$$

Using Lemma 4.3, we also have

$$\begin{aligned} \|x_k - \phi(y_{k+1})\| &\leq L_\phi \|(x_k, y_{k+1}) - \zeta\| \\ &\leq L_\phi (\|x_k - \zeta_x\| + |y_{k+1} - \zeta_y|) \\ &\leq 2L_\phi \|(x_k, y_k) - \zeta\|, \end{aligned}$$

from which follows:

$$\|x_{k+1} - \phi(y_{k+1})\| \leq \frac{4\bar{\gamma}_x}{\psi(u_x)} L_\phi^2 \|(x_k, y_k) - \zeta\|^2.$$

Lemma 4.3 then gives $\|\phi(y_{k+1}) - \zeta_x\| \leq l_\phi |y_{k+1} - \zeta_y|$, from which we deduce:

$$\begin{aligned} \|x_{k+1} - \zeta_x\| &\leq \|x_{k+1} - \phi(y_{k+1})\| + \|\phi(y_{k+1}) - \zeta_x\| \\ &\leq \left(\frac{4\bar{\gamma}_x}{\psi(u_x)} L_\phi^2 + l_\phi L_y \right) \|(x_k, y_k) - \zeta\|^2. \end{aligned} \tag{37}$$

Combining (36) and (37) gives us:

$$\|(x_{k+1}, y_{k+1}) - \zeta\| \leq \|x_{k+1} - \zeta_x\| + |y_{k+1} - \zeta_y| \leq L_x \|(x_k, y_k) - \zeta\|^2,$$

and using (A_{19}) concludes the proof of this part.

4.8. **Proof of Part (b) of Theorem 4.1.** We let $k := K$, from Corollary 4.11 and Lemma 4.7 (applied with $(c, b) = (z'_k, y_{k+1})$):

$$\begin{aligned} |y_{k+1} - \zeta_y| \leq 3\frac{m - l'}{m} \Big(&(1 + (\bar{\mu}\bar{e})^m) \beta_{m,l'}(h(z'_k, .); y_{k+1}) \\ &+ L_{\bar{e}}^{m/(m-l')} \|z'_k\|^{1/(m-l')} \Big). \end{aligned}$$

Then, using part (a) of Lemma 4.13 and the fact that $\mathcal{F}^z_{m,l,l'}(x_K, y_K) = 0$, we deduce inequality (23) of the theorem.

From Corollary 4.12, we know $\zeta_k \in \bar{B}(\zeta_y, r_y)$ and therefore, Lemma 4.7 (applied with $(c, b) = (z'_k, \zeta_k)$) gives:

$$\beta_{m,l'}(h(z'_k, .); \zeta_k) \leq \frac{\beta_{m,l'}(h(0, .); \zeta_k) + L_{\bar{e}}^{m/(m-l')} \|z'_k\|^{1/(m-l')}}{1 - (\bar{\mu}\bar{e})^m}.$$

Using again $\mathcal{F}^z_{m,l,l'}(x_K, y_K) = 0$, we deduce:

$$\beta_{m,l'}(h(z'_k, .); \zeta_k) \leq \frac{\beta_{m,l'}(h(0, .); \zeta_k) + L_{\bar{e}}^{m/(m-l')} \mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1})/\mathcal{G}_z}{1 - (\bar{\mu}\bar{e})^m}. \tag{38}$$

In a similar way, using Lemma 4.7 with $(c, b) = (z'_k, \zeta_y)$, we obtain:

$$\begin{aligned}\beta_{m,l'}(h(z'_k, .); \zeta_y) &\le \frac{\beta_{m,l'} + L_{\bar{e}}^{m/(m-l')} \|z'_k\|^{1/(m-l')}}{1 - (\bar{\mu}\bar{e})^m} \\ &\le \frac{\beta_{m,l'} + L_{\bar{e}}^{m/(m-l')} \mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1})/\mathcal{G}_z}{1 - (\bar{\mu}\bar{e})^m}. \qquad (39)\end{aligned}$$

On the other hand, by Corollary 4.12 and (A_{12}) one has

$$\gamma_m(h(0, .); \zeta_k)|\zeta_k - \zeta_y| \le u,$$

hence Proposition 2.2 yields, via (A_{16}):

$$\beta_{m,l'}(h(0, .); \zeta_k) \le \frac{1-u}{\psi_m(u)} \left(\beta_{m,l'}(1-u)^{\frac{l'+1}{m-l'}} + (2m-1)|\zeta_k - \zeta_y| \right). \qquad (40)$$

From part (a) of Lemma 4.13, we have:

$$\mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1}) \le \Xi \beta_{m,l'}(h(z'_k, .); \zeta_k).$$

Then, from this inequality and successively using (38), (40), (28) and (39), we deduce:

$$\mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1}) \le \Xi(T_1 \beta_{m,l'} + T_2/\mathcal{G}_z \mathcal{B}_{m,l'}(h(z'_k, .), y_k; y_{k+1})),$$

whence (24). The proof of Theorem 4.1 is now completed.

4.9. **Algorithm from Estimates at the Initial Point.** In this last subsection, we describe how valid input quantities for Theorem 4.1 can be computed from estimates at the initial point only. The strategy is the same as in Section 1.4: we perform cluster location first with Theorem 3.1, then deduce upper bounds on point estimates in the cluster in order to enter the approximation algorithm. For the sake of simplicity, the algorithm presented below assumes that $\mathtt{f}$ and $\mathtt{g}$ are polynomial maps. Nevertheless this algorithm could be extended to broader classes of maps for which all the necessary point estimates are computable (as we did in [12, Section 6] for the univariate case). In Appendix B, we provide the reader with all the remaining technical details concerning the computations in practice, and we report on numerical experiments.

The initial point is still written (x_0, y_0). We assume that the location criterion underlying Theorem 3.1 holds. Namely, we assume that there exists a cluster of $m - l'$ zeros of $(\mathtt{f}, \mathtt{g}^{[l']})$ in $\bar{B}((x_0, y_0), r)$. From these data, we attempt to apply Theorem 4.1, as explained in the following algorithm. Of course, the process breaks as soon as a computation is *not possible* or a *requirement* fails. From now, ζ represents a point of the cluster.

- At (x_0, y_0), compute upper bounds γ_{x_0}, δ_{x_0} and σ_{x_0} on $\gamma(\Sigma; x_0, y_0)$, $\|D\Sigma(x_0, y_0)\|$ and $\|D\Sigma(x_0, y_0)^{-1}\|$ respectively. Use Proposition 1.15 to compute (if possible) upper bounds γ_x, δ_x, σ_x of $\gamma(\Sigma; \zeta)$, $\|D\Sigma(\zeta)\|$ and $\|D\Sigma(\zeta)^{-1}\|$, respectively (hence (A_1) is satisfied).
- At (x_0, y_0) compute suitable values for $\lambda_{\mathtt{g},x_0}$ and $\rho_{\mathtt{g},x_0}$ in order to have $[\mathtt{g} - \mathtt{g}(x_0, y_0)]_{(x_0,y_0)} \le \lambda_{\mathtt{g},x_0} t/(1 - \rho_{\mathtt{g},x_0} t)$. Use Corollary 1.6 to compute (if possible) λ and ρ satisfying (A_2).
- Compute λ_Φ, ρ_Φ, λ, ρ, require (A_3) and compute l_ϕ, L_ϕ and r_x.

- Use Proposition 1.15 to compute (if possible) upper bounds $\bar{\gamma}_x$ and $\bar{\delta}_x$ of $\max(\gamma(\Sigma; x, y) \mid (x, y) \in \bar{B}(\zeta, l_\phi r))$ and $\max(\|D\Sigma(x, y)\| \mid (x, y) \in \bar{B}(\zeta, l_\phi r))$. According to Lemma 4.3, conditions (A_4) and (A_5) are satisfied. Invoking this lemma is legitimate since it only makes use of (A_1), (A_2) and (A_3). This is the reason why we have introduced (A_3) in the statement of Theorem 4.1 although it is a consequence of (A_9).
- Compute u_x, require (A_6) and compute L_z, r_z.
- Compute x'_0, z'_0.
- Compute the series expansion of $h(z'_0, .)$ at y_0 at precision $2(m - l')$ and, then, upper bounds on $\beta_{m,l'}(h(z'_0, .); y_0)$, $m!/|D_y^m h(z'_0, y_0)|$. From λ, ρ and using Corollary 1.6, compute (if possible) a geometric majorant series $\lambda' t/(1 - \rho' t)$ of $h - h(z'_0, y_0)$ at (z'_0, y_0). Then use Proposition B.3 to compute an upper bound on $\gamma_m(h(z'_0, .); y_0)$ (we set the parameter i to $2(m - l')$ in our program).
- Use λ', ρ' and Proposition 2.6 to compute (if possible) upper bounds on:
$$\max(\beta_{m,l'}(h(z, .); y_0) \mid z \in \bar{B}(z'_0, \|z'_0\| + r_z)),$$
$$\max(\gamma_m(h(z, .); y_0) \mid z \in \bar{B}(z'_0, \|z'_0\| + r_z)),$$
$$\max(m!/|D_y^m h(z, y_0)| \mid z \in \bar{B}(z'_0, \|z'_0\| + r_z)).$$
- Use Proposition 2.2 and the previous quantities to compute (if possible) an upper bound on
$$3\frac{m - l'}{m} \max(\beta_{m,l'}(h(z, .); y) \mid z \in \bar{B}(z'_0, \|z'_0\| + r_z),\ y \in \bar{B}(y_0, 2r))$$
and take r_y as the maximum of r and the latter upper bound, so that conditions (A_7) and (A_8) are satisfied.
- Compute $\bar{r}$ and require (A_9).
- Use Proposition 2.2 to compute valid values for $\bar{\beta}_m$, $\bar{\gamma}_m$ and $\bar{\sigma}_m$, in order to satisfy (A_{10}), (A_{11}) and (A_{12}).
- Compute $\bar{\lambda}$, $\bar{\rho}$, $\bar{\mu}$, $L_{\bar{e}}$, $\bar{e}$ and require (A_{13}).
- Require (A_{14}) and (A_{15}).
- Compute u, v, $\bar{u}$ and require (A_{16}).
- Compute $\mathcal{C}$, $\mathcal{G}_y$, $\bar{\mathcal{C}}$, $\underline{\kappa}$, $\bar{\kappa}$, χ, Ξ, T_1, T_2, T_β, $\mathcal{G}_z$, T_y and require (A_{17}).
- Compute $L_{y,1}$, $L_{y,2}$, L_y and require (A_{18}).
- Compute L_x and require (A_{19}).

It is straightforward to check that, if ζ is an isolated zero of multiplicity m then, for any $l \in \{0, \dots, m-1\}$, $l' \le l$, this algorithm works with $(x_0, y_0) := \zeta$ and $r = 0$ (in particular this implies $\bar{r} = r_x = r_y = r_z = 0$). By continuity, we deduce that if (x_0, y_0) is sufficiently close to ζ and r sufficiently small then the algorithm also works. By deformation, it follows that the algorithm actually locates and approximates clusters of embedding dimension 1.

For clusters with positive diameter, the iteration stops. At the end, Theorem 4.1 asserts that y_{K+1} is close to ζ_y at a distance bounded in terms of $\beta_{m,l'}(h(0, .); \zeta_y)$. According to [12, Theorem 2.1], this quantity can be bounded in terms of the diameter of the cluster of zeros of $h(0, .)$ (if the diameter of the cluster is sufficiently small). As for the x coordinates, no such result actually holds. Nevertheless, one can iterate Newton's operator

$x \mapsto N(\mathtt{f}(., y_{K+1}); x)$ from x_{K+1} to improve the x coordinates. The convergence of this iteration can be quantified by means of Lemma 4.4 and the iteration can be stopped as soon as it reaches a distance to $\phi(y_{K+1})$ which is about $\mathcal{B}_{m,l'}(h(z'_K, .), y_K; y_{K+1})$. We leave out the details here.

In Appendix B we report on numerical experiments, that all confirm the expected theoretical behaviors. We also exhibit examples for which the eight possible cases of the approximation algorithm actually occur in practice.

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Appendix A. Local Compositional Inverse

In this appendix, we study the behavior of majorant series under local compositional inversion. The following results generalize and slightly improve those of Dedieu *et al.* in [8, Section 3]. We also give different and simpler proofs. We start with the same proposition:

Proposition A.1. [8, Theorem 3.2] *Let f be an analytic map from an open neighborhood U of a in $\mathbb{C}^n$ to $\mathbb{C}^n$ such that $Df(a) = \mathrm{Id}$. Then f is invertible in a neighborhood of $f(a)$. Let $H \in t^2\mathbb{R}\{t\}$ be such that $[f - f(a)]_a \leq t + H$. Let $F := t - H$, and F^{-1} denote the compositional inverse of F, then $\left[f^{-1} - a\right]_{f(a)} \leq F^{-1}$.*

Proof. We refer to the proof of [8, Theorem 3.2], based on Faà di Bruno's formula. □

By considering inverses on the left and inverses on the right, we now relax the condition $Df(a) = \mathrm{Id}$.

Corollary A.2. *Let f be an analytic map from an open neighborhood U of a in $\mathbb{C}^n$ to $\mathbb{C}^n$. Assume that $Df(a)$ is invertible, then f is invertible in a neighborhood of $f(a)$. Let g_1 and g_2 be defined by*

$$g_1(x) := f(a + Df(a)^{-1}x) - f(a), \quad g_2(x) := Df(a)^{-1}(f(a + x) - f(a)).$$

For $i \in \{1, 2\}$, let $H_i \in t^2\mathbb{R}\{t\}$ satisfy $[g_i]_0 \leq t + H_i$. Define $F_i := t - H_i$. Then

$$\left[Df(a)(f^{-1} - a)\right]_{f(a)} \leq F_1^{-1}(t),$$
$$\left[f^{-1} \circ Df(a) - a\right]_{Df(a)^{-1}f(a)} \leq F_2^{-1}(t).$$

Hence

$$\left[f^{-1} - a\right]_{f(a)} \leq \|Df(a)^{-1}\| F_1^{-1}(t) \text{ and } \left[f^{-1} - a\right]_{f(a)} \leq F_2^{-1}(\|Df(a)^{-1}\|t).$$

Proof. Since $Dg_i(0) = \mathrm{Id}$, $i \in \{1, 2\}$, we apply the previous proposition with g_i and H_i at 0. Then we use the fact that

$$f^{-1}(y) = a + Df(a)^{-1}g_1^{-1}(y - f(a)) = a + g_2^{-1}(Df(a)^{-1}(y - f(a))). \quad \square$$

Now we specialize this last corollary to the special case of geometric majorant series and obtain:

Corollary A.3 (for geometric majorant series)**.** *Let f be an analytic map from an open neighborhood U of a in $\mathbb{C}^n$ to $\mathbb{C}^n$. Assume that $Df(a)$ is invertible.*

a. *If $[f - f(a)]_a \le \lambda t/(1-\rho t)$ then*

$$\left[Df(a)(f^{-1} - a)\right]_{f(a)} \le \frac{t}{1 - \vartheta\rho\|Df(a)^{-1}\|t}.$$

b. *If $\left[Df(a)^{-1}(f - f(a))\right]_a \le \dfrac{\lambda\|Df(a)^{-1}\|t}{1-\rho t}$ then*

$$\left[f^{-1} \circ Df(a) - a\right]_{Df(a)^{-1}f(a)} \le \frac{t}{1-\vartheta\rho t}.$$

In both cases, ϑ denotes the largest root of $P(x) := 1 - 2(1+2\|Df(a)^{-1}\|\lambda)x + x^2$.

Observe that the defining polynomial P for ϑ admits two nonnegative real roots and that ϑ is larger than one.

Proof. For part (a), we apply the previous corollary with $H_1 = \mu\bar\rho t^2/(1-\bar\rho t)$, where $\mu = \|Df(a)^{-1}\|\lambda$ and $\bar\rho = \rho\|Df(a)^{-1}\|$. We are then led to study the inverse G of

$$F = t - \frac{\mu\bar\rho t^2}{1-\bar\rho t}.$$

This inverse is given by the explicit formula

$$G = \frac{1 + \bar\rho t - \sqrt{1 - 2(1+2\mu)\bar\rho t + \bar\rho^2 t^2}}{2\bar\rho(1+\mu)}.$$

The rest of the proof is thus concentrated in the study of one specific map, which is performed in Lemma A.5 below.

For part (b) we apply the previous corollary with $H_2 = \mu\rho t^2/(1-\rho t)$ and the rest is similar to part (a). □

Before coming to technical lemmas, we give two examples illustrating the sharpness of the previous corollary.

First, we consider $f = t - t^2/(1-t)$. Then part (a) or (b) of the corollary applies with $\lambda = \rho = f'(0) = 1$ and gives the bound $\left[f^{-1}\right]_0 \le t/(1-\vartheta t) =: B(t)$, with $\vartheta = 3 + 2\sqrt{2}$. The value of ϑ in the denominator is optimal. Indeed, from the explicit formula for G above, we can use Darboux's theorem [19] to deduce that as n tends to infinity, the nth coefficient of the Taylor expansion of G at the origin behaves like $\kappa\vartheta^n n^{-3/2}$ for some constant κ. A smaller constant $\vartheta' < \vartheta$ in the denominator of B would lead to an asymptotic bound of order ϑ'^n, incompatible with the actual behavior. Concerning the numerator of B, it is also optimal since the first coefficient of f^{-1} is 1.

Our second example is $f = t - 2t^2/(1-t)$. As in the previous example, $\rho = f'(0) = 1$, but now, $\lambda = 2$. The value of ϑ given by part (a) is $5 + 2\sqrt{6} \approx 9.90$, which again is optimal with respect to the asymptotic behavior of the Taylor coefficients, while the numerator is again sharp for the first coefficient. Applying part (b) yields $\gamma(f^{-1}; 0) \le (3 + 2\sqrt{2})2 \approx 11.66$, which illustrates the gain in dealing with a majorant geometric series given by two real numbers instead of only the gamma estimate.

Let us now complete the proof of Corollary A.3. The following lemma is probably classical, although we could not find a reference to it.

Lemma A.4. *Let $P(t) := \sum_{i=0}^{p} a_i t^i$ be a polynomial with real coefficients such that $a_0 > 0$ and the Taylor series of $1/P$ at the origin has nonnegative coefficients. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying the inequality*

$$\mathcal{R}(u;n) := \sum_{i=0}^{p} a_{p-i} u_{n+i} \leq 0, \qquad n \geq 0. \tag{41}$$

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence defined by $v_i = u_i$, for $0 \leq i < p$ and $\mathcal{R}(v;n) = 0$ for $n \geq 0$. Then $u_n \leq v_n$ for all $n \geq 0$.

Proof. Let $U(t) := \sum u_n t^n$ (resp. $V(t) := \sum v_n t^n$) be the generating series of the sequence $(u_n)_{n\in\mathbb{N}}$ (resp. $(v_n)_{n\in\mathbb{N}}$). By multiplying (41) by t^{n+p} and summing over n, we get an inequality

$$P(t)U(t) \leq P(t)V(t).$$

Observe that $P(t)V(t)$ is a polynomial of degree at most $p-1$. Since $1/P$ has nonnegative coefficients, part (d) of Proposition 1.1 allows to multiply both sides of the inequality by $1/P$. Extracting coefficients of t^n on both sides then concludes the proof. □

Lemma A.5. *Let $P(t) = 1 - 2\nu t + t^2$, with $\nu > 1$ and $G(t) = \dfrac{1 + t - \sqrt{P(t)}}{1+\nu}$. Then*

$$[G]_0 \leq \frac{t}{1 - \vartheta t},$$

where ϑ is the largest root of P.

Proof. The function G satisfies the linear differential equation

$$P(t)G'(t) + (\nu - t)G(t) = 1 - t.$$

It follows that the sequence $(u_n)_{n\in\mathbb{N}}$ of its Taylor coefficients at the origin satisfies the linear recurrence equation

$$u_n = \left(2 - \frac{3}{n}\right)\nu u_{n-1} - \left(1 - \frac{3}{n}\right) u_{n-2}, \qquad n \geq 3, \tag{42}$$

with initial conditions $u_0 = 0$, $u_1 = 1$, $u_2 = (\nu - 1)/2$. We now prove that

$$0 < u_n < 2\nu u_{n-1} - u_{n-2}, \qquad n \geq 4. \tag{43}$$

To this aim, we first prove by induction that $u_n > u_{n-1}$ for $n \geq 3$. For $n = 3$, we have $u_3 = \nu u_2$ from (42) which gives the desired inequality since $\nu > 1$ and $u_2 > 0$. For $n \geq 3$, let $v_n = u_n/u_{n-1}$ and assume $v_{n-1} > 1$, then (42) implies

$$v_n = \left(2 - \frac{3}{n}\right)\nu - \frac{1 - 3/n}{v_{n-1}} \geq 1 + (2 - 3/n)(\nu - 1) > 1.$$

From $u_n/u_{n-1} > 1$, the sign of u_n follows (u_2 being positive) and since $\nu > 1$, we also get $v_n < 2\nu - 1/v_{n-1}$ for $n \geq 4$, whence the second part of the desired inequality (43). The polynomial P can be written $(1 - \vartheta t)(1 - t/\vartheta)$

with ϑ its largest positive root. It follows that the Taylor series of $1/P$ at the origin has nonnegative coefficients. We can therefore apply the previous lemma to the sequence u_{n+2}, which gives

$$\frac{G(t)-u_0-u_1t}{t^2} \leq \frac{u_2+(u_3-2\nu u_2)t}{P(t)}.$$

Isolating $G(t)/t$ and computing a partial fraction expansion of the right-hand side gives

$$\frac{G(t)}{t} \leq 1+\frac{\nu(1-\nu)}{2}+\frac{C(\vartheta)}{1-\vartheta t}+\frac{C(1/\vartheta)}{1-t/\vartheta},$$

with $C(\vartheta)=\frac{\nu-1}{2}\,(\nu-\vartheta/2)$. Now, obviously (since $\vartheta>1$)

$$\frac{1}{1-t/\vartheta}-1 \leq \frac{\vartheta^{-2}}{1-\vartheta t}-\vartheta^{-2},$$

which makes it possible to bound the latter summand in terms of the second one. A straightforward computation gives $C(\vartheta)+C(1/\vartheta)/\vartheta^2=(\nu-1)/(2\vartheta)$. The proof is concluded by showing that this last quantity is smaller than 1. Indeed, by writing $\vartheta=\nu+\sqrt{\nu^2-1}$ and dividing by $\nu-1$, we see that $\vartheta/(\nu-1)>2$, which is sufficient. The numerator 1 is then dictated by the first coefficient $u_1=1$. □

Appendix B. Numerical Experiments

We relate numerical experiments with the location algorithm of Section 3, and with the approximation algorithm of Section 4.9. Before all, we need to explain the basic devices used in our program. We consider the following examples parameterized by the real positive number N. For the sake of simplicity we restrict to polynomial maps.

Example 1: $\mathtt{f}_1(x_1,y):=x_1+5x_1^2$, $\mathtt{g}(x_1,y):=(y^m-10^{-mN})(1-y^m)$. Here $n=2$ and $(\mathtt{f},\mathtt{g})$ admits a cluster of m zeros in a neighborhood of the origin, which collapses to the origin when N tends to infinity.

Example 2: $\mathtt{f}_1(x_1,x_2,y):=10^{-4N}+3x_1+5x_2+y+x_2y^2+6x_1^3x_2+5x_1x_2y^2-5x_1y^3-x_1y^4$, $\mathtt{f}_2(x_1,x_2,y):=10^{-4N}-x_1+2x_2+x_1y-x_1x_2y-x_1y^3-4x_2y^3+3x_1x_2^3y$, $\mathtt{g}(x_1,x_2,y):=10^{-4N}+x_1+11/2x_1^2-2x_2+1419x_2^3+y^3$. Here $n=3$ and $(\mathtt{f},\mathtt{g})$ admits a cluster of 4 zeros around the origin, which tends to a multiple zero when N goes to infinity.

Computations are performed with the Maple computer algebra system version 7. The Digits environment variable controls the number of decimal digits that Maple uses when calculating with software long floating-point numbers. Heuristically, in order to avoid rounding off problems, we set this variable to $2mN$. We leave this precision problem here for the sake of simplicity.

B.1. **Approximation of Point Estimates.** We describe the formulas we use in our implementation for computing upper bounds on point estimates.

B.1.1. *Computing δ_x and σ_x.* At a given point (x_0, y_0), we first address the problem of computing $\|D\Sigma(x_0, y_0)\|$ and $\|D\Sigma(x_0, y_0)^{-1}\|$. Both problems correspond to computing norms of matrices M of the form

$$M = \begin{pmatrix} \mathrm{Id} & A \\ 0 & 1 \end{pmatrix},$$

where A is a column vector of length $n-1$, we use the following classical formula:

Proposition B.1. [21, Exercise 6.10, p. 116]

$$\|M\|^2 = 1 + \frac{1}{2}\|A\|^2 + \|A\|\sqrt{1 + \frac{1}{4}\|A\|^2}.$$

B.1.2. *Majorant Series.* Computing majorant series of polynomials reduces to upper bounding the norms of all its derivatives. For this purpose, we make use of the norm $\|.\|_\infty$, as defined just below. More sophisticated devices could be used but we retained this one for efficiency and simplicity reasons.

Let $E := \{e_1, \ldots, e_n\}$ denote the canonical basis of $\mathbb{C}^n$, $A \in \mathcal{L}_l(\mathbb{C}^n; \mathbb{C}^s)$, it is fast to compute the norm $\|.\|_\infty$, defined by

$$\|A\|_\infty := \max_{u_1 \in E, \ldots, u_l \in E} \|A u_1 \ldots u_l\|.$$

Then, we content ourselves with the following upper bound on the norm of A:

Lemma B.2. $\|A\| \leq n^l \|A\|_\infty$.

Proof. Let $A_{i_1,\ldots,i_l} := A e_{i_1} \ldots e_{i_l}$. Let $u_1, \ldots, u_l$ be unit vectors of $\mathbb{C}^n$, we compute:

$$\begin{aligned} \|A u_1 \ldots u_l\| &= \left\| \sum_{i_1=1}^{n} \cdots \sum_{i_l=1}^{n} A_{i_1,\ldots,i_l} u_{1,i_1} \cdots u_{l,i_l} \right\| \\ &\leq n^l \|A\|_\infty, \end{aligned}$$

where $u_{i,j}$ denotes the jth coordinate of u_i. □

In particular, since the map Σ of Theorems 3.1 and 4.1 is polynomial, we use the following upper bound:

$$\gamma(\Sigma; a, b) \leq \max_{l \geq 2} \left(n^l \left\| D\Sigma(a,b)^{-1} \frac{D^l \Sigma(a,b)}{l!} \right\|_\infty \right)^{\frac{1}{l-1}}.$$

Concerning the computations of λ_{g} and ρ_{g} at a given point (a, b) such that $[\mathsf{g} - \mathsf{g}(a,b)]_{(a,b)} \leq \lambda_{\mathsf{g}} t / (1 - \rho_{\mathsf{g}} t)$, we arbitrarily take:

$$\lambda_{\mathsf{g}} := \max_{l \geq 1} \frac{n^l}{l!} \|D^l \mathsf{g}(a,b)\|_\infty, \;\; \rho_{\mathsf{g}} := 1.$$

m	2	2	2	3	4
N	5	10	20	10	10
$\|(x_0, y_0)\|$	$1.34\ 10^{-6}$	$1.34\ 10^{-6}$	$1.34\ 10^{-6}$	$2.69\ 10^{-6}$	$3.37\ 10^{-7}$
r_x^-	$6.09\ 10^{-5}$	$3.63\ 10^{-5}$	$3.63\ 10^{-5}$	$2.08\ 10^{-5}$	$1.45\ 10^{-5}$
r_y^-	$5.95\ 10^{-5}$	$3.52\ 10^{-5}$	$3.52\ 10^{-5}$	$1.88\ 10^{-5}$	$1.43\ 10^{-5}$

TABLE 5. Cluster location with Example 1

m	4	4	4
N	5	10	20
$\|(x_0, y_0)\|$	$1.32\ 10^{-5}$	$1.32\ 10^{-5}$	$1.32\ 10^{-5}$
r_x^-	$1.15\ 10^{-4}$	$1.15\ 10^{-4}$	$1.15\ 10^{-4}$
r_y^-	$9.15\ 10^{-5}$	$9.15\ 10^{-5}$	$9.15\ 10^{-5}$

TABLE 6. Cluster location with Example 2

B.1.3. *Upper Bounds on γ_m.* The last basic computation we deal with is the computation of upper bounds on γ_m. The following proposition quantifies how such a bound can be determined from a geometric series majoration, exploiting the possible knowledge of a series expansion.

Proposition B.3. *Let q denote a one complex variable function, let m be an integer such that $q^{(m)}(z) \neq 0$, let $\sigma_m \geq m!/|q^{(m)}(z)|$, and let λ, ρ be nonnegative real numbers such that $[q - q(z)]_z \leq \lambda t/(1-\rho t)$. Let $i \geq m+1$, and let p denote the unique polynomial of degree at most $i-1$ such that $q - p \in \mathcal{O}_z((x-z)^i)$ then*

$$\gamma_m(p; z) \leq \gamma_m(q; z) \leq \max\left(\gamma_m(p; z),\ \rho\left(\sigma_m \lambda \rho^{m-1}\right)^{\frac{1}{i-m}}\right).$$

Proof. By construction, we have $\sigma_m \lambda \rho^{m-1} \geq 1$, hence

$$\sup_{j\geq i}(\sigma_m \lambda \rho^{j-1})^{\frac{1}{j-m}} = \sup_{j\geq i} \rho(\sigma_m \lambda \rho^{m-1})^{\frac{1}{j-m}} = \rho(\sigma_m \lambda \rho^{m-1})^{\frac{1}{i-m}}. \qquad \square$$

B.2. **Cluster Location.** In Tables 5 and 6, we relate numerical experiments with Theorem 3.1: the parameter l is set to 0 and (x_0, y_0) is computed as $(\exp(\imath\pi/4), \dots, \exp(\imath\pi/4))$ times the largest negative power of 2 that satisfies the conditions of the theorem. Here $\imath \in \mathbb{C}$ represents the square root of -1 with positive imaginary part. We indicate the values r_x^- and r_y^- (as defined in Theorem 3.1): recall that $\bar{B}(x_0, r_x^-) \times \bar{B}(y_0, r_y^-)$ contains a cluster of m zeros.

We observe that the location process does not depend much on the size of the cluster. As expected, the location becomes the more especially difficult as the cardinality of the cluster increases.

We compute the requested upper bound on γ_m by using Proposition B.3 at (z_0', y_0) with $i := 2m$, and by using the geometric series majoration given in (21). Finally, the series expansion of $h(z_0', .)$ at y_0 is directly computed

N	10	20	40	80
$\|(x_0, y_0)\|$	$2.05\ 10^{-11}$	$2.05\ 10^{-11}$	$2.05\ 10^{-11}$	$2.05\ 10^{-11}$
r_y	$1.96\ 10^{-8}$	$1.39\ 10^{-8}$	$1.39\ 10^{-8}$	$1.39\ 10^{-8}$
$\mathcal{G}_y$	$1.05\ 10^{5}$	$1.05\ 10^{5}$	$1.05\ 10^{5}$	$1.05\ 10^{5}$
$\mathcal{G}_z$	$6.80\ 10^{3}$	$6.79\ 10^{3}$	$6.79\ 10^{3}$	$6.79\ 10^{3}$
K	0	1	2	3
$\mathcal{F}^y_{m,l,l'}$	$-$	$1, -$	$1, +, -$	$1, 1, -, -$
$\mathcal{F}^z_{m,l,l'}$	0	$1, 0$	$1, 1, 0$	$1, 1, 1, 0$
$\|x_{K+1}\|$	$1.05\ 10^{-21}$	$5.60\ 10^{-42}$	$1.57\ 10^{-82}$	$1.23\ 10^{-163}$
$\|y_{K+1}\|$	$1.45\ 10^{-11}$	$6.87\ 10^{-30}$	$3.24\ 10^{-48}$	$2.92\ 10^{-98}$
$\mathcal{B}_{m,l'}(y_{K+1})$	$1.00\ 10^{-10}$	$1.00\ 10^{-20}$	$1.00\ 10^{-40}$	$1.00\ 10^{-80}$

TABLE 7. Cluster approximation with Example 1 and $m = 2$

N	10	20	40	80
$\|(x_0, y_0)\|$	$1.02\ 10^{-11}$	$1.02\ 10^{-11}$	$1.02\ 10^{-11}$	$1.02\ 10^{-11}$
r_y	$3.90\ 10^{-8}$	$2.62\ 10^{-8}$	$2.62\ 10^{-8}$	$2.62\ 10^{-8}$
$\mathcal{G}_y$	$1.20\ 10^{6}$	$1.19\ 10^{6}$	$1.19\ 10^{6}$	$1.19\ 10^{6}$
$\mathcal{G}_z$	$9.59\ 10^{4}$	$9.49\ 10^{4}$	$9.49\ 10^{4}$	$9.49\ 10^{4}$
K	0	1	2	3
$\mathcal{F}^y_{m,l,l'}$	∞	$1, \infty$	$1, \infty, \infty$	$1, +, \infty, \infty$
$\mathcal{F}^z_{m,l,l'}$	0	$1, 0$	$1, 1, 0$	$1, 1, 1, 0$
$\|x_{K+1}\|$	$2.64\ 10^{-22}$	$3.50\ 10^{-43}$	$6.13\ 10^{-85}$	$1.88\ 10^{-168}$
$\|y_{K+1}\|$	$7.27\ 10^{-12}$	$2.59\ 10^{-47}$	$2.03\ 10^{-56}$	$1.17\ 10^{-153}$
$\mathcal{B}_{m,l'}(y_{K+1})$	$1.00\ 10^{-10}$	$9.99\ 10^{-21}$	$1.00\ 10^{-40}$	$1.00\ 10^{-80}$

TABLE 8. Cluster approximation with Example 1 and $m = 4$

from the one of $\Phi(z_0', .)$, which is obtained by means of the classical symbolic Newton iteration.

This location process gives us a ball $\bar{B}((x_0, y_0), r)$ that contains the cluster, when taking $r := \sqrt{(r_x^-)^2 + (r_y^-)^2}$. With this data in hand, we can now enter the approximation algorithm described in Section 4.9, on which we report experiments in the following last subsection.

B.3. **Cluster Approximation.** In Tables 7, 8 and 9, we report on numerical experiments with the approximation algorithm presented in Section 4.9. Here the notation is the one used in Section 4.9. We take $l' := l := 0$ and, for different values of m and N, we compute (x_0, y_0) as the vector $(\exp(\imath\pi/4), \dots, \exp(\imath\pi/4))$ times the largest negative power of 2 that does not provoke an error in the whole algorithm. We indicate the values of the

N	10	20	40	80
$\|(x_0, y_0)\|$	$2.52\ 10^{-11}$	$2.52\ 10^{-11}$	$2.52\ 10^{-11}$	$2.52\ 10^{-11}$
r_y	$1.91\ 10^{-8}$	$1.19\ 10^{-8}$	$1.19\ 10^{-8}$	$1.19\ 10^{-8}$
$\mathcal{G}_y$	$2.04\ 10^{5}$	$2.04\ 10^{5}$	$2.04\ 10^{5}$	$2.04\ 10^{5}$
$\mathcal{G}_z$	$2.69\ 10^{5}$	$2.69\ 10^{5}$	$2.69\ 10^{5}$	$2.69\ 10^{5}$
K	0	1	1	2
$\mathcal{F}^y_{m,l,l'}$	$-$	$1, -$	$1, +$	$1, 1, +$
$\mathcal{F}^z_{m,l,l'}$	0	$0, 0$	$0, 0$	$0, 0, 0$
$\|x_{K+1}\|$	$2.95\ 10^{-12}$	$7.46\ 10^{-25}$	$4.76\ 10^{-50}$	$1.93\ 10^{-100}$
$\|y_{K+1}\|$	$1.45\ 10^{-11}$	$3.67\ 10^{-24}$	$2.34\ 10^{-49}$	$9.52\ 10^{-100}$
$\mathcal{B}_{m,l'}(y_{K+1})$	$9.59\ 10^{-11}$	$9.59\ 10^{-21}$	$9.59\ 10^{-41}$	$9.59\ 10^{-81}$

TABLE 9. Cluster approximation with Example 2

main parameters, namely r_y, $\mathcal{G}_y$ and $\mathcal{G}_z$. Then we give the number K of iterations, the sequences of flags $(\mathcal{F}^y_{m,l,l'}(x_k, y_k))_k$ and $(\mathcal{F}^z_{m,l,l'}(x_k, y_k))_k$, the norms of the last iterate, and the value of $\mathcal{B}_{m,l'}(h(z'_K, .), y_K; y_{K+1})$, abbreviated $\mathcal{B}_{m,l'}(y_{K+1})$.

In all our examples the diameter of the cluster is about 10^{-N}, as confirmed by the values of $\mathcal{B}_{m,l'}(h(z'_K, .), y_K; y_{K+1})$. In all cases, the x coordinates are already close to the cluster without performing the post-treatment mentioned at the end of Section 4.9. Lastly, it is important to observe that all the eight possible cases of the algorithm actually occur in practice.

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Index of Main Symbols

$[f]_a$, 11
$\alpha(f;a)$, 5
$\alpha_m(h;a)$, 5
$\beta(f;a)$, 5
$\beta_m(h;a)$, 5
$\gamma(f;a)$, 5
$\gamma_m(h;a)$, 5
$\theta_{m,l,\delta}$, 29
$\Sigma(\mathtt{f},x_0,y_0;x,y)$, 4
$\tau_{m,l,0}(v)$, 21
$\tau_{m,l,1}(v)$, 21
$\phi(\mathtt{f},x_0,y_0;y)$, 4
$\Phi(\mathtt{f},x_0,y_0;z,y)$, 4
$\psi(u)$, 5
$\psi_m(u)$, 5
$B(\zeta,r)$, 3
$\bar{B}(\zeta,r)$, 3
$\mathcal{B}_{m,l}(f,x_0;x_1)$, 21
$C_{m,l,l',\delta}(u)$, 29
$\mathtt{f}$, 2
$f^{[l]}$, 4
$\mathcal{F}^y_{m,l,l'}(x_k,y_k)$, 29
$F(f,a;t)$, 13
$\mathcal{F}^z_{m,l,l'}(x_k,y_k)$, 30
$\mathtt{g}$, 2
$\mathtt{g}^{[l]}$, 4
$\mathcal{G}_y$, 29
$\mathcal{G}_z$, 29
$h(f,x_0,y_0;z,y)$, 4
$N(f;x)$, 5
$N_m(h;x)$, 5
$N_{m,l,l'}(x_k,y_k)$, 29
r_y, 29
$u_{m,l,\delta}$, 29
$v_{m,l}$, 21

Symbols defined in Tables 1 and 3 do not appear in this index.

Marc Giusti, Laboratoire STIX, École polytechnique, 91128 Palaiseau, France
E-mail address: Marc.Giusti@polytechnique.fr

Grégoire Lecerf, Laboratoire de Mathématiques, Université de Versailles Saint-Quentin-en-Yvelines, 45 avenue des États-Unis, 78035 Versailles, France
E-mail address: Gregoire.Lecerf@math.uvsq.fr

Bruno Salvy, Projet ALGO, INRIA Rocquencourt, 78153 Le Chesnay, France
E-mail address: Bruno.Salvy@inria.fr

Jean-Claude Yakoubsohn, Laboratoire MIP, Bureau 131, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France
E-mail address: yak@mip.ups-tlse.fr